First-Price Auctions in Online Display Advertising

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Abstract

We link the rapid and dramatic move from second-price to first-price auction format in the display advertising market to the move from the waterfalling mechanism employed by publishers for soliciting bids in a pre-ordered cascade over exchanges, to an alternate header bidding strategy that broadcasts the request for bid to all exchanges simultaneously. First, we argue that the move by the publishers from waterfalling to header bidding was a revenue improving move for publishers in the old regime when exchanges employed second-price auctions. Given the publisher move to header bidding, we show that exchanges move from second-price to first-price auctions to increase their expected clearing prices. Interestingly, when all exchanges move to first-price auctions, each exchange faces stronger competition from other exchanges and some exchanges may end up with lower revenue than when all exchanges use second-price auctions; yet, all exchanges move to first-price auctions in the unique equilibrium of the game. We show that the new regime hinders the exchanges' ability to differentiate in equilibrium. Furthermore, it allows the publishers to achieve the revenue of the optimal mechanism despite not having direct access to the advertisers.

1 Introduction

Advertising via online display ads has seen a dramatic rise in the last decade. When display ads are generated by publishers, they are typically sold via ad-exchanges to advertisers that bid in real time in spot auctions. In the last five years, most of these exchanges moved from a second-price to a first-price auction format. In this paper, we show that this move can be attributed to another recent change in this industry regarding publishers' bid solicitation mechanism. Publishers moved from a sequential bid solicitation, known as waterfalling, to a parallel one, known as header bidding, in which they send the request-for-bid to all exchanges simultaneously. Using a parsimonious model of this ecosystem, we show that the publisher's move from waterfalling to header bidding can cause the subsequent move by the exchanges from second-price to first-price auctions. In addition to providing a new explanation for the change in auction format, our analysis suggests that this change reduces the ability of exchanges to differentiate themselves and lowers the fees they are able to extract from advertisers, while allowing the publisher to achieve maximum revenue despite the advertisers being fragmented among multiple parallel exchanges.

Evolution of the Display Advertising Market

Display advertising, with an estimated market share of 54% in the US, has grown to be a significant proportion of the digital advertising market.[1] The early promise of digital advertising came from search advertising that allowed advertisers to find customers at a deeper stage in their purchase funnel and also validate their interest by requiring payments only for clicks. However, as adoption of mobile devices grew and video content became more popular than before, a large volume of user attention became available in the form of user visits to web-sites and mobile apps other than search. This has led to an increase in the availability of user eyeballs, referred to as *impressions*, in websites and mobile apps visited by users. Furthermore, new technology in display advertising, e.g., real-time bidding (RTB), has allowed advertisers to target consumers dynamically and at an individual level, and has made display advertising more appealing than before.

Early methods of selling display ads involved more traditional channels with a salesforce and the sale of fixed large inventories of eyeballs, and were the mainstay of companies that ran publishing networks that produced such large streams of impressions (such as the Yahoo and Microsoft Publishing Networks). After traditional sales, the uncertain inventory of impressions that were unsold, the so-called ‘remnants’, were then auctioned off in real time in one or multiple marketplaces, called *exchanges*. In such real-time bidding (RTB) auctions, when such impressions became available, the publisher generated a request for bids dynamically and tried to sell the impression to the highest bidder. Given the lower volume of these available remnants, early publisher networks that processed remnant inventory preferred to send the request for bids to only a few reliable large advertising networks or exchanges so as to get a quick and reasonably high bid for the impression. Typically,

[1] https://forecasts-na1.emarketer.com/584b26021403070290f93a56/5851918a0626310a2c1869ca

such preferred exchanges were ordered in sequence of their expected price they fetched per impression (eCPM), and the real-time bidding system would sequentially go down this order of exchanges, generating a request, waiting for a short while before timing out and moving to the next exchange in the sequence. The first acceptable bid in the order was accepted, and auction reserve prices (often called "floor prices" in this context) were used to control the level of acceptability. This form of real-time bidding that evolved in the early days of display advertising was termed *waterfalling*.[2]

One of the undesirable features of the waterfall model is that it creates a fractured market among the ad exchanges leaving advertisers in a quandary about which exchanges to associate with to spend their budgets most effectively. In particular, the advertiser's decision of which exchange to join should take into account both its order in the waterfall and the competition within that exchange. For publishers, this format can lead to the loss of high-valued advertisers in the later stages of the waterfall whose bids are never considered.

Given these inefficiencies, around 2014, a new format for requesting bids from ad exchanges called *header bidding* was introduced by publishers. In this format, rather than go through different partner ad exchanges in sequence, the publisher broadcasts the request for bid simultaneously to all ad exchanges and after collecting all returned bids within a reasonable time, picks the best one.[3] Since its introduction, header bidding caught on very rapidly and became the mainstream format of publishers by the end of 2016. By some estimates, the percentage of the top publishers that used header bidding increased from 0% to over 70% in the period 2014 to 2016.[4]

Until header bidding was introduced in the display advertising marketplace, the auction format for selling display ads was the well established second-price format (with a potential reserve price set by the publisher), that the industry inherited from the paid search advertis-

[2]Zawadziński, Maciej (2018) "Waterfalling, Header Bidding and New Auction Dynamics", https://clearcode.cc/blog/sequential-auctions-header-bidding-first-price-second-price-auctions/

[3]For this reason, this was also called advance or pre-bidding.

[4]https://www.businessinsider.com/header-bidding-gains-momentum-drives-up-publisher-ad-revenue-2016-5

ing world. However, in early 2017, right after the introduction of header bidding, several ad exchanges started experimenting with a first-price auction format instead. This move came about in a variety of ways, including the introduction of "soft floors" which were set by the ad exchanges. While the publisher supplied a reserve price with the request for bids called the "hard floor", each ad exchange would set another higher value as a soft floor and change the rule of the local auction in the following way: If there were at least two bids above the soft floor, they participated in a regular second price auction; with only one bid above the soft floor, the soft floor now served as the clearing price; with all bids below the soft floor but some still above the hard floor, the bids participated in a first-price auction. Note that by setting the soft floor sufficiently high, the auction format is effectively converted from a second-price to a first-price auction. Indeed, several exchanges such as AppNexus advised advertisers to bid in soft-floor auctions as they bid in first-price auctions.[5] The lack of transparency about the values of the soft floors set in these auctions led to such intermediate formats being quickly replaced by the more transparent first-price format with a reserve price.[6] After Google's move to first-price auctions in 2019, all major exchanges now use first-price auctions to sell display advertising impressions, when a publisher sends the request for bid to multiple exchanges.

We emphasize that the move from second-price to first-price auctions happened in situations when a publisher sends the request for bid to multiple third-party exchanges (typically through header bidding), and allocates the impression to the exchange with the highest clearing price, e.g., when exchanges such as Rubicon[7], Pubmatic[8], and OpenX compete with each other to sell the publisher's impression.[9] This is in contrast with situations where the publisher sells its inventory directly to advertisers without calling third-party exchanges

[5] https://www.linkedin.com/pulse/things-you-should-know-sspexchange-auctions-today-paul-gubbins/

[6] Shterev, Boris (2020), "First-Price Auction? What does it mean for publishers?" https://www.pubgalaxy.com/first-price-auction-what-does-it-mean-for-publishers/

[7] https://rubiconproject.com/insights/thought-leadership/greater-transparency-choice-in-auction-dynamics/

[8] https://community.pubmatic.com/display/SSP/Support+for+First-Price+Auctions+FAQs

[9] https://docs.openx.jp/demandpartners/first-price-auctions.html

(e.g., when Google sells YouTube impressions), or uses only a single exchange to sell the impressions. Indeed, as observed in Tunuguntla and Hoban (2021), second-price auctions remain the standard when the publisher sells the impressions exclusively through one exchange, and, as such, the exchange is the final arbiter of impression placement (e.g., scenarios when OpenX runs the final auction[10]).

Research Questions

Our primary research question is regarding the rapid transition of exchanges from second to first price auction format, and why this move occurred at this late stage rather than with the initial advent of display advertising.

A second research question is the consequence of this move from second to first price auctions for the strategies of the publishers and the advertisers. Publishers will have to re-evaluate their floor prices as a consequence of this double move from waterfalling to header bidding and from second-price to first-price auctions. Similarly, advertisers will have to modify their bidding strategies from the old regime to the new. In addition, given the change in the marketplace rules, they may also need to re-evaluate whether the choice of ad exchanges with which they affiliate themselves (often at non-negligible costs) is optimal and worth the corresponding fees.

A third research question involves how these two changes in the marketplace (from waterfall to header bidding, and of the auction format from second to first price) affect ad exchange revenues, in both the short and the long term.

Contributions

In this paper, we study the above research questions using a simple model of the display advertising ecosystem. Our model involves a single publisher, two ad exchanges, and a minimal set of (four) advertisers who decide to affiliate with one of the ad exchanges by

[10] https://docs.openx.jp/demandpartners/first-price-auctions.html

paying their respective fees. We begin our analysis in the old regime where the ad exchanges use second-price auctions with reserve prices, and the publisher uses waterfalling. In this waterfalling setting, we show (in Proposition 1) that the revenue of the ad exchanges, and the advertisers' utilities are not affected by the auction format, so there is no incentive for the exchanges to change from their historically prevalent second-price format. We note in Propositon 2 that in this setting, the ad exchanges are able to set nonzero entry fees for the advertisers.

Our next result (Proposition 3) shows that when the ad exchanges use a second-price auction, moving from waterfalling to header bidding increases the publisher revenue thus providing a simple economic explanation for this initial move of the publishers.

To answer the primary research question of the subsequent move of the ad exchanges from second to first-price auctions, we analyze the choice of auction format by the exchanges after the publisher's move to header bidding. Our main result is a new explanation in Proposition 4, where we show that under header bidding, there is a unique equilibrium in which both exchanges run first-price auctions. We can contrast this to our earlier observation (Proposition 1) that under waterfalling, the revenues of the ad exchanges are not affected by the auction format. In this way, we show that the move to first-price auctions might be a direct economic consequence of the widespread adoption of header bidding by publishers.

This result provides an alternate reason to the main explanation that has been advanced so far for this change in auction format in the literature (e.g., Akbarpour and Li (2020)), which argues that the move is the result of trust issues because advertisers do not have to trust the exchange in a first-price auction. Our model gives an intuitive explanation for the rapid adoption of first-price auction by exchanges that occurred soon after the exponential growth of header bidding.[11] As additional evidence of the plausibility of our explanation, we note that Google adopted the first-price auction format for its exchange platform, where it acts as an intermediary and the participating publishers use header bidding, but has retained

[11] https://adexchanger.com/online-advertising/google-switches-to-first-price-auction/

the second-price auction format for the sales of its own inventory (e.g., YouTube) where it assumes the role of the publisher and there are no intermediaries.[12]

Next, we show that, under header bidding with first-price auctions, the exchanges' equilibrium fees for the advertisers become zero (Proposition 5). In other words, under waterfalling, the exchanges can differentiate themselves based on their positions in the waterfall sequence, and under second-price auctions, the exchanges can differentiate based on the set of advertisers they are affiliated with. This led to nonzero fees for ad exchanges under waterfalling with second-price auctions (Proposition 2). However, the combination of header bidding and first-price auctions removes the exchanges' ability to differentiate based on their set of advertisers or their position, and lowers their equilibrium buyer-side fees. This finding is consistent with recent reductions in exchange fees from an average of 25% in 2016 to around 15% in 2018, and predicted to be in single digits in near future.[13]

From a managerial point of view, our results shed light on how the new selling mechanism, i.e., the combination of header bidding and first-price auctions, affect the strategies of advertisers, publishers, and exchanges. We show that while advertisers should shade their bids in first-price auctions, they should bid as if all advertisers (from all exchanges) are in the same auction. In other words, under header bidding with first price auctions, each advertiser is directly competing with all advertisers from all exchanges. For publishers, we show that the new mechanism greatly simplifies the reserve price optimization problem. Furthermore, by setting the reserve prices optimally, publishers can achieve the revenue of the optimal mechanism (Myerson, 1981), even though they do not have direct access to advertisers. Finally, since the new mechanism eliminates the exchanges' ability to differentiate based on the number of their advertisers and their position in the waterfall, exchanges have to devise new differentiation strategies in order to survive in the long run.

The rest of this paper is structured as follows. First, we review the related literature. In

[12] https://www.blog.google/products/admanager/simplifying-programmatic-first-price-auctions-google-ad-manager/

[13] https://adexchanger.com/platforms/rubicon-project-eliminates-buy-side-fees/ and https://adexchanger.com/platforms/big-changes-coming-auctions-exchanges-roll-dice-first-price/

Section 2, we present the model. In Section 3, we analyze the model and discuss the results. We conclude the paper in Section 4. All proofs are relegated to the Appendix.

Related Literature

Our work is related to the growing literature on online advertising auctions. Katona and Sarvary (2010) and Jerath et al. (2011) study advertisers' incentives in obtaining lower vs. higher positions in search advertising auctions. Sayedi et al. (2014) investigate advertisers' poaching behavior on trademarked keywords, and their budget allocation across traditional media and search advertising. Desai et al. (2014) analyze the competition between brand owners and their competitors on brand keywords. Lu et al. (2015) and Shin (2015) study budget constraints, and budget allocation across keywords. Zia and Rao (2017) look at the budget allocation problem across search engines. Wilbur and Zhu (2009) find the conditions under which it is in a search engine's interest to allow some click fraud. Cao and Ke (2019) and Jerath et al. (2018) study manufacturer and retailers' cooperation in search advertising and show how it affects intra- and inter-brand competition. Amaldoss, Desai and Shin (2015) show how a search engine can increase its profits and also improve advertisers' welfare by providing first-page bid estimates. Berman and Katona (2013) study the impact of search engine optimization, and Amaldoss, Jerath and Sayedi (2015) analyze the effect of keyword management costs on advertisers' strategies. Katona and Zhu (2017) show how quality scores can incentivize advertisers to invest in their landing pages and to improve their conversion rates. Long et al. (2018) study the informational role of search advertising on the organic rankings of an online retail platform. Our work is different from these papers as we study display advertising auctions in real-time bidding. In our paper, the auctioneer (i.e., the exchange mechanism) is different from the publisher, whereas in search advertising models, the publisher (i.e., the search engine) also designs the auction.

Our work contributes to the vast literature on display advertising. Empirical works in this area have assessed the effectiveness of display advertising in various contexts. Lambrecht and Tucker (2013) demonstrate that retargeting may not be effective when consumers have not adequately refined their product preferences. Hoban and Bucklin (2015) find that display advertising increases website visitations for a large segment of consumers along the purchase funnel, but not for those who had visited before. Bruce et al. (2017) examine the dynamic effects of display advertising and show that animated (vs. static) ads with price information are the most effective in terms of consumer engagement. Rafieian and Yoganarasimhan (2018) study the role of targeting in online advertising and shows that ad networks may benefit from preserving customer privacy. Rafieian (2019) shows that publishers can improve their revenue by optimally sequencing the ads that they show to a customer in a session. On the theoretical front, Sayedi et al. (2018) study advertisers' bidding strategies when publishers allow advertisers to bid for exclusive placement on the website. Zhu and Wilbur (2011) and Hu et al. (2015) study the trade-offs involved in choosing between "cost-per-click" and "cost-per-action" contracts. Berman (2018) explores the effects of advertisers' attribution models on their bidding behavior and their profits. Despotakis et al. (2021) and Gritckevich et al. (2018) look at how ad blockers affect the online advertising ecosystem, and Dukes et al. (2019) show how skippable ads affect publishers' and advertisers' strategies as well as their profits. Kuksov et al. (2017) study firms' incentives in hosting the display ads of their competitors on their websites. Choi and Sayedi (2019) study the optimal selling mechanism when a publisher does not know, but benefits from learning, the performance of advertisers' ads. These papers, unlike ours, do not study the roles of intermediaries (i.e., exchange platforms) in the market. In contrast, the focus of our research is to study what triggered the intermediaries' move from second-price to first-price auctions, and how this move affects publishers and advertisers in this market.

In the context of real-time bidding auctions, Johnson (2013) estimates the financial impact of privacy policies on publishers' revenue and advertisers' surplus. Rafieian (2020) characterizes the optimal mechanism when the publisher uses dynamic ad sequencing. Zeithammer (2019) shows that introducing a soft reserve price, a bid level below which a winning bidder pays his own bid instead of the second-highest bid, cannot increase publishers' revenue in RTB auctions when advertisers are symmetric; however, it can increase the revenue when advertisers are asymmetric. The model in Zeithammer (2019) has only one exchange, and, therefore, cannot distinguish between header bidding and waterfalling. We show that while under waterfalling the results of Zeithammer (2019) continue to hold, under header bidding first-price auctions generate a higher revenue for the publisher than second-price auctions even with symmetric advertisers. Sayedi (2018) analyzes the interaction between selling impressions through real-time bidding and selling through reservation contracts; it shows that, in order to optimize their revenue, publishers should use a combination of RTB and reservation contracts. In Sayedi (2018), there is only one exchange, and header bidding (compared to waterfalling) affects how advertisers in RTB compete with those in reservation contracts. Choi and Mela (2018) study the problem of optimal reserve prices in the context of RTB, and, using a series of experiments, estimate the demand curve of advertisers as a function of the reserve price. Since the dataset in Choi and Mela (2018) is from 2016, exchanges still use second-price auctions. We show that when exchanges use first-price auctions, the publisher's problem of reserve price optimization becomes much simpler. Choi et al. (2017) provide an excellent summary of the literature and key trends in the area of display advertising markets. They also mention the move of ad exchanges from second-price to first-price auctions to be related to header bidding in that it enables the highest bidder to win (which is not necessarily the case with the second-price format), but do not provide a model or analysis. Interestingly, they leave it to future research to analyze the impact of the recent changes in selling mechanisms on advertisers' and publishers' revenues, which is a gap we attempt to fill with our work.

2 Model

There is one publisher who is selling an impression, two exchanges, and four advertisers who can bid for the impression through one of the exchanges. Each advertiser's valuation for the impression is an i.i.d. draw from a uniform distribution on the interval $[0, 1]$, with cumulative distribution function $F(x) = x$.

Exchanges. Exchanges are intermediaries that connect publishers to advertisers. The revenue of an exchange comes from *buyer-side fees*, i.e., how much they charge advertisers for their service, and *seller-side fees*, how much they charge publishers for their service. We assume that Exchange i, $i \in \{1, 2\}$, sets a fee $f_i \geq 0$ for advertisers who want to use its platform. In practice, the buyer-side fee can have a complex structure and be a combination of advertisers' bidding and winning volumes, as well as their average submitted bids.[14] In the interest of parsimony, we assume that an advertiser has to pay a flat fee f_i if it wants to use Exchange i, where f_i is set by the exchange. Given the fees, and other parameters that we will discuss shortly, the advertisers decide which exchange to join. We use n_i to denote the number of advertisers that use Exchange i. The exchanges also charge publishers a seller-side fee. Seller-side fees are negotiated between exchanges and publishers, and are typically a fraction of an exchange's contribution to the publisher's revenue. In our model, we assume that the publisher pays fraction f of the revenue that it collects through Exchange i to Exchange i. For example, if an exchange sells the impression of the publisher for a price of 1, the publisher keeps $1 - f$, and gives f to the exchange. To facilitate exposition, we assume that seller-side fee f is the same for both exchanges and exogenous in the model.[15] If Exchange i sells the impression at price p, its total revenue is $n_i f_i + fp$, and if it does not

[14] It has even been reported that many exchanges were not transparent in terms of what fees they charged advertisers, and advertisers were in many cases surprised when they realized that some of payments were being paid to the exchange instead of the publisher. For example, see https://adexchanger.com/ad-exchange-news/explainer-widespread-fee-practice-behind-guardians-lawsuit-vs-rubicon-project/.

[15] Note that, even though the exchanges are ex-ante symmetric, they can create value (and differentiate) by offering additional bids from new advertisers. In other words, the publisher is not constrained to work with only one exchange, and benefits from allowing as many exchanges as possible, as long as each exchange brings new advertisers to the game. This allows the exchanges to set positive seller-side fees.

sell the impression, its total revenue is $n_i f_i$.

In addition to setting buyer-side fees, the exchanges also decide what auction format to use. An exchange can use a first-price auction or a second-price auction to sell the publisher's impression on its platform. The exchange uses the format that maximizes its revenue; the format of the auction is revealed to the advertisers before they submit their bids. In both formats, the highest bidder wins as long as the bid is greater than or equal to the reserve price. The clearing price of a second-price auction is the maximum of the second-highest bid and the reserve price. The clearing price of a first-price auction is the highest bid. If no one bids the reserve price or higher, in both auction types, the clearing price is zero.

Publisher. When an impression arrives (i.e., a consumer visits the publisher's website or app), the publisher sends a 'request for bid' to the exchanges. The publisher can send the request for bids simultaneously to both exchanges, or send them sequentially. As we discussed in the introduction, the sequential strategy is called "waterfalling" and the simultaneous one is called "header bidding". Under waterfalling, the publisher waits for the outcome of the first exchange, and if the impression is sold in the first exchange (i.e., there is at least one bid greater than or equal to the reserve price), the publisher does not send it to the second exchange. If the impression is left unsold in the first exchange, the publisher sends it to the second exchange. The publisher can also choose the order of the exchanges, i.e., to which exchange to send the impression first; without loss of generality, we assume that, if the publisher uses waterfalling, it sends the impression to Exchange 1 first. Under header bidding, the publisher sends the impression to both exchanges at the same time. Each exchange runs an auction and sends its clearing price back to the publisher; the publisher selects the exchange with the highest clearing price as long as at least one of the clearing prices is greater than zero. If both clearing prices are zero, i.e., the impression is left unsold in both exchanges, the impression remains unallocated, and the publisher's revenue becomes zero.

When RTB started, waterfalling was the only strategy available to the publishers. In

2014, some publishers moved to header bidding, and by the end of 2016, more than 70% of top publishers in the US were using header bidding. Even though the choice of header bidding versus waterfalling is a publisher's decision, in our model, we analyze the two models separately. This allows us to highlight how the publisher's move from waterfalling to header bidding triggered the adoption of first-price auctions by exchanges, and explain how this market has evolved over time. Finally, the publisher sets reserve prices for each exchange. We use r_i to denote the reserve price of Exchange i. In practice, and also in our model, optimizing the reserve prices is an essential part of revenue optimization for publishers in RTB markets (Choi and Mela, 2018). If the impression is allocated to Exchange i, with clearing price p, the publisher's revenue is $(1-f)p$.

Advertisers. There are $n = 4$ advertisers in our model. While we can prove many of our results for larger number of advertisers, having $n = 4$ has two benefits for us. First, the number is large enough so that if two advertisers join each exchange, we still have within-exchange competition between advertisers in both exchanges. Furthermore, we can prove the results analytically in the case of $n = 4$, while finding advertisers' bidding strategies for larger values of n becomes analytically intractable.[16] Advertisers make two decisions in our model. First, they decide which exchange to join, if any.[17] This decision happens after the advertisers learn the buyer-side fees, f_1 and f_2, but before they learn their valuation for the impression. In practice, advertisers choose an exchange before impressions arrive; the partnership between exchanges and advertisers is usually long-term, and advertisers cannot switch exchanges in real time, before or after every impression. Therefore, advertisers have to use the valuation distribution, as opposed to the actual realization of the valuation, when deciding which exchange to join. An advertiser can also decide to not join either of the two exchanges (e.g., if the fees are too high). In that case, the advertiser's utility becomes zero, i.e., the advertiser "leaves the game."

[16]In Section C.3 we numerically verify our results for larger numbers of advertisers.

[17]While it is possible for an advertiser to bid in several exchanges at the same time, this is not commonly observed in practice. Some potential reasons may be to avoid paying access fees to multiple platforms and also the potential risks of indirectly competing against itself from different exchanges.

When an impression arrives at an exchange, advertisers in that exchange decide how much to bid for the impression. At this time, each advertiser knows his private value for the impression, an i.i.d. draw from the uniform $U[0,1]$ distribution. The advertiser also knows the reserve prices, r_1 and r_2, and the format of the auction in both exchanges, i.e., whether each exchange uses first-price or second-price auctions. The advertiser, however, does not know other advertisers' valuation for the impression (but only their distribution). These assumptions are consistent with what advertisers know when submitting their bids in this market. Note that an advertiser's bid is only submitted to the exchange that he has joined. Under header bidding, winning in the affiliated exchange does not imply that the advertiser will get the impression, as the impression may be allocated to the winner of the other exchange. If an advertiser with valuation v, who uses Exchange i, is allocated an impression at clearing price p, the advertiser's utility is $v - p - f_i$. If the advertiser does not win, his utility is $-f_i$.

2.1 Timeline

Here is a summary of the timeline of the game. To highlight the similarities and for succinctness, we present both the waterfalling and header-bidding scenarios of the publisher in the same outline below.

1. Exchanges decide their buyer-side fees f_1 and f_2.
2. Advertisers choose which exchange to join (if any).
3. The publisher sets reserve prices r_1 and r_2 for the two exchanges.
4. Exchanges decide their auction format, i.e., whether to use a second-price auction or a first-price auction.
5. Advertisers' valuations are privately realized; they submit their bids to their affiliated exchanges.

6. • Under Waterfalling:

   (a) Exchange 1 runs its auction. If the clearing price p_1 of the auction is larger than zero (i.e., at least one bidder bids at least the reserve price), then the publisher allocates the impression to the winner of Exchange 1 for price p_1 and the game ends.

   (b) If the clearing price in Exchange 1 is zero (i.e., no bidder in Exchange 1 bids at least the reserve price), then the publisher moves to Exchange 2. The second exchange runs its auction with reserve price r_2 and sends the clearing price p_2 to the publisher. If $p_2 > 0$ (i.e., there is at least one bid greater than or equal to r_2), the publisher allocates the impression to the winner of Exchange 2 for price p_2. Otherwise, the impression remains unsold.

   • Under Header Bidding:

   (a) The publisher sends the impression to both exchanges; the exchanges run their auctions simultaneously. They send their clearing prices p_1 and p_2 to the publisher.

   (b) If $\max(p_1, p_2) > 0$ (i.e., in at least one exchange one bidder bids greater than or equal to the reserve price of that exchange), the publisher allocates the impression to the exchange with the higher clearing price[18] at that exchange's clearing price. Otherwise, the impression is left unsold.

Note that, in our timeline, the exchanges decide the auction format after the advertisers join exchanges and the publisher sets the reserve prices. This is because in practice, during the transition from second-price to first-price auctions, some exchanges (such as Rubicon) announced that they would decide the auction format in real time, after an impression arrives.[19] While all exchanges eventually moved to a pure first-price auction format (as it

[18] In case of a tie between the two exchanges, the winner is selected randomly with probability $\frac{1}{2}$.

[19] https://www.linkedin.com/pulse/things-you-should-know-sspexchange-auctions-today-paul-gubbins/

happens in our model as well), from a modeling perspective, we have taken into account that they had the option of choosing a different format for every impression.

Rather than model the industry transition as above, an alternate motivation may be to analyze the current situation in the market where the exchanges commit to an auction format and announce it before the reserve prices are set. Motivated by this, and to verify the robustness of our results, in Sections A.2.1 and A.2.2 we consider two alternative timelines where exchanges decide the auction formats before the publisher sets the reserve prices. In Section A.2.1 advertisers join exchanges before the exchanges decide their auction format, while in Section A.2.2 advertisers join exchanges after the exchanges decide their auction format and before the publisher sets the reserve prices. We show that our main results continue to hold under both these extensions.

3 Analysis

We use backward induction to solve the game under each scenario, waterfalling and header bidding, separately. To keep the flow of this section consistent with the evolution of the display advertising industry, we start by analyzing the waterfalling game.

3.1 Waterfalling

In the last stage of the game, advertisers have to decide how much to bid for the impression. Suppose that there are n_1 and n_2 advertisers in the first and the second exchange and the reserve prices are set to r_1 and r_2 respectively.[20] The following lemma summarizes the advertisers' bidding strategies.

Lemma 1. *Under waterfalling, if Exchange i is using a second-price auction with reserve r_i, all advertisers bid truthfully.*[21] *If the exchange is using a first-price auction with reserve*

[20]Note that $n_1 + n_2$ can be less than 4, as some advertisers may not join any of the exchanges.

[21]Bidding "truthfully" here means that the advertisers bid their true valuations (e.g., there is no bid shading). We borrow this term from the mechanism-design literature (e.g., see Krishna, 2009). Note that

r_i, *an advertiser with valuation* $v \geq r_i$ *bids*

$$v - \frac{v}{n_i} + \frac{r_i^{n_i}}{n_i \cdot v^{n_i - 1}}.$$

Lemma 1 summarizes the advertisers' bids in first-price and second-price auctions as a function of the reserve price and the number of advertisers. The expressions that we have for the bids are the standard expressions for first-price and second-price auctions with reserve prices (e.g., see Krishna, 2009). The lemma shows that, under waterfalling, existence of Exchange 1 does not directly affect the bids of Exchange 2 and vice versa. In other words, advertisers in different exchanges do not directly compete with each other under waterfalling. These advertisers, however, compete with each other indirectly by how the publisher sets the reserve prices. In particular, since the publisher knows that it can sell the impression in Exchange 2 if it is left unsold in Exchange 1, the publisher sets a higher reserve price for Exchange 1 than if Exchange 2 did not exist. As such, the existence of an advertiser in Exchange 2 increases the expected payment of an advertiser in Exchange 1, and the existence of an advertiser in Exchange 1 lowers the probability of winning for an advertiser in Exchange 2. In the following lemma, we show how the optimal reserve prices are set by the publisher under both first-price and second-price auctions.

Lemma 2. *Under waterfalling, regardless of the auction format (i.e., in both second-price and first-price auctions), the optimal reserve price of Exchange 2 is* $\frac{1}{2}$ *and the optimal reserve price of Exchange 1 is*

$$1 - \frac{1}{n_2 + 1} + \frac{1}{(n_2 + 1) \cdot 2^{n_2 + 1}}.$$

Lemma 2 has two interesting implications. First, it shows that the optimal reserve prices are not affected by the format of the auction. In fact, as we later show in Proposition 1, the revenues of the publisher and both exchanges are the same in first-price auctions as in

even in mechanisms where advertisers do not bid truthfully, e.g., a first-price auction, the advertisers' true types may still be revealed in equilibrium.

second-price auctions. The second implication of Lemma 2 is regarding the value of the optimal reserve prices. As we can see, the optimal reserve price of the second exchange is $\frac{1}{2}$, regardless of the number of advertisers in each exchange. Under waterfalling, r_2 matters only when the impression is left unsold in Exchange 1. Therefore, if the publisher does not sell the impression in Exchange 2, it will generate 0 revenue. This reduces the revenue maximization of the publisher for Exchange 2 to a standard optimal auction setting (with no "outside option" for the seller). In fact, the optimal reserve price $\frac{1}{2}$ in Lemma 2 is Myerson's optimal reserve price for the case of uniform value distributions for the bidders (Myerson, 1981).[22]

We can see from Lemma 2 that the optimal reserve price of Exchange 1 is always greater than or equal to $\frac{1}{2}$, and only a function of n_2, the number of advertisers in Exchange 2. Intuitively, when the publisher is setting the reserve price of Exchange 1, it has to take its expected revenue from Exchange 2 into account. The expected revenue from Exchange 2 acts as an "outside option" for the publisher when selling its impression in Exchange 1, i.e., the publisher's revenue if the impression is left unsold in Exchange 1. As n_2 increases, the publisher's expected revenue, thus the value of "keeping" the impression, from Exchange 1 increases; therefore, the optimal reserve price of Exchange 1 increases as n_2 increases.

In the next proposition, we compare the revenue of the publisher and the exchanges under first-price auctions to their revenues under second-price auctions.

Proposition 1. *Under waterfalling, the revenue of the publisher, the revenue of the exchanges, and the advertisers' utilities are not affected by the auction format.*

Proposition 1 shows that, under waterfalling, if an exchange moves from second-price to first-price auction, or vice versa, the move does not affect the revenue of the publisher or either of the exchanges. This result follows from the *revenue equivalence principle* (e.g., see Krishna, 2009). Basically, if an exchange changes its auction format from second-price to

[22] Myerson's optimal reserve price when the CDF and PDF of bidders' valuation are F and f, respectively, is $\phi^{-1}(0)$ where $\phi(x) = x - \frac{1-F(x)}{f(x)}$.

first-price, advertisers who now have to pay what they bid (rather than the next highest bid) shade their bids. While the exchange (and the publisher) make more revenue from a given set of bids, the amount by which the advertisers lower their bids cancels out the publisher's extra revenue from a given set of bids. This is in fact a general result about symmetric bidders, and is not driven by our assumption about the number of advertisers, or the advertisers' valuations being uniformly distributed.

Proposition 1 explains why exchanges did not move to first-price auctions under waterfalling. When real-time bidding was introduced in 2009, exchanges used the second-price auction, an already popular auction format in the context of online search advertising. Advertisers were already familiar with second-price auctions, and the truthful nature of the auction made bidding strategies relatively simple. Since, under waterfalling, first-price auctions are equivalent to second-price in terms of expected equilibrium revenue, exchanges had no reason to abandon the simple and already accepted second-price auctions. It was only in 2017, after header bidding became widely popular among publishers, that some exchanges started experimenting with first-price auctions, and eventually moved to the first-price auction format. Indeed, later in Proposition 4, we show that under header bidding, first-price auctions and second-price auctions are not equivalent anymore.

Next, we analyze the advertisers' choices of exchanges, and the fees that the exchanges set in equilibrium. For any given fees f_1 and f_2, the advertisers' choices of exchanges can have multiple equilibria. In the following proposition, we show that the exchanges can charge the advertisers a positive fee in at least some equilibria of the game.[23]

Proposition 2. *Under waterfalling, the exchanges can obtain positive buyer-side revenue in equilibrium; i.e., there are equilibria in which* $n_1 f_1 + n_2 f_2 > 0$.

Proposition 2 shows that, under waterfalling, the exchanges can obtain positive revenue by charging advertisers a type of fee that is referred to as buyer-side fee in this industry. In

[23] In Section C.1 (Web Appendix), we prove that under some mild assumptions, the exchanges obtain positive total buyer-side revenue in *all* equilibria of the game.

fact, before the move to first-price auctions, exchanges had been charging, and in many cases increasing, their buyer-side fees. Interestingly, in Proposition 5, we show that under header bidding with first-price auctions, the exchanges' ability to obtain positive buyer-side revenue disappears; i.e., both exchanges charge zero fees, i.e., $f_1 = f_2 = 0$ in *all* equilibria of the game. Intuitively, the reason that exchanges can set positive buyer-side fees in equilibrium under waterfalling is that the exchanges can differentiate in their offerings to the advertisers. In particular, as we demonstrate in the following example, the order of the exchanges in the waterfall sequence, and the number of advertisers that each exchange has, can make one exchange more attractive to the advertisers than the other.

Example 1. There is an equilibrium where the fees are $f_1^* = 0$, $f_2^* = \frac{n\cdot(4^n-1)-(2^n-1)^2-n^2\cdot 2^n}{n^2(n+1)\cdot 2^{2n+1}}$, and all advertisers join Exchange 2. Each advertiser's expected utility in this equilibrium is $\frac{1}{n(n+1)} - \frac{1}{n\cdot 2^n} + \frac{1}{(n+1)\cdot 2^{n+1}} - f_2^*$; the publisher's expected revenue is $(1-f)\cdot\left(1 - \frac{2}{n+1} + \frac{1}{(n+1)\cdot 2^n}\right)$. The revenue of Exchange 1 is 0, and the expected revenue of Exchange 2 is $nf_2^* + f \cdot \left(1 - \frac{2}{n+1} + \frac{1}{(n+1)\cdot 2^n}\right)$.[24]

In the equilibrium in Example 1, even though Exchange 1 has no fee, all advertisers join Exchange 2 that has a positive fee. Intuitively, if an advertiser switches to Exchange 1, it will face a very high reserve price because the publisher has a high expected revenue from Exchange 2 (as discussed in Lemma 2). As such, advertisers are better off paying fee f_2 and staying with Exchange 2 than deviating to Exchange 1. In other words, under waterfalling, exchanges with more advertisers are able to set a higher buyer-side fee.[25] As we see later in Section 3.2, this advantage disappears when exchanges move to header bidding with first-price auctions.

Before concluding this section, and as a segue into the discussion of header bidding, we present the following proposition. This proposition examines an off-equilibrium path

[24] We can prove that this equilibrium has the highest expected publisher revenue, as well as the highest total exchange revenue among all equilibria of the game.

[25] In Section C.4, we show a similar result when the decision for the ordering of the exchanges under waterfalling is endogenous in the model.

subgame solution that is meant to reflect the state of the industry just before header bidding was deployed by publishers and to model their consideration in switching from waterfalling to header bidding. It describes a specific situation where exchanges use the second-price auction format that was widely popular earlier and advertisers cannot move between exchanges for exogenous reasons such as inertia.

Proposition 3. *Assuming that the advertisers have chosen their exchanges and the exchanges use second-price auctions, for any* $n_1, n_2 > 0$*, the revenue of the publisher when using header bidding is higher than when using waterfalling.*

Proposition 3 shows that the publisher's revenue under header bidding is higher than under waterfalling, if the exchanges keep using second-price auctions. We emphasize, however, that this result is off the equilibrium path and applies to the sub-game where advertisers have already joined exchanges; i.e., as we show later in Section 3.2, under header bidding, both exchanges use first-price auctions in equilibrium. Our goal here is to show that the publishers' move to header bidding was not necessarily motivated by the exchanges' subsequent move to first-price auctions. In fact, there is no evidence that publishers anticipated the adoption of first-price auctions by exchanges when they moved to header bidding; and, for almost a year after the publisher's move to header bidding, the exchanges continued to run second-price auctions. Proposition 3 explains the rapid growth of header bidding during a time when exchanges were still using second-price auctions.[26]

3.2 Header Bidding

In this section, we analyze the advertisers', the exchanges', and the publisher's strategies under header bidding. As in Section 3.1, we use backward induction to solve the game. As before, we assume that n_1 and n_2 advertisers have joined Exchanges 1 and 2, respectively, and

[26] We note that Sayedi (2018) has a similar result in a model with no exchanges and two horizontally differentiated advertisers.

the reserve prices are r_1 and r_2. The following lemma summarizes the advertisers' bidding strategies in first-price and second-price auctions.

Lemma 3. *Under header bidding, if an exchange uses a second-price auction, all advertisers within that exchange bid truthfully. If Exchange i uses a first-price auction, assuming that $r_1 = r_2 = \frac{1}{2}$, the bid of an advertiser with valuation v who uses Exchange i is as follows.*

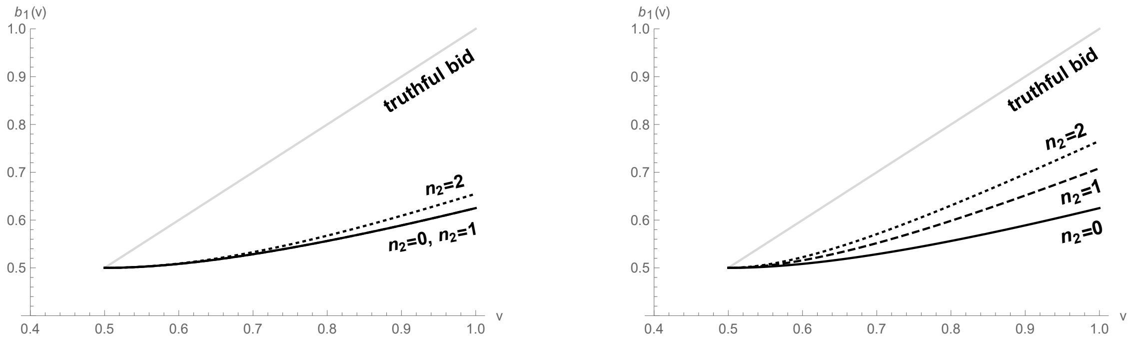
- *If both exchanges use first-price auctions:*

$$v - \frac{v}{n_1 + n_2} + \frac{1}{(n_1 + n_2) \cdot 2^{n_1+n_2} \cdot v^{n_1+n_2-1}}.$$

- *If the other exchange uses a second-price auction, the advertisers' bid function, which varies depending on specific values of n_1 and n_2, is presented in the Appendix.*

Note that we have stated the second part of Lemma 3, when at least one exchange uses a first-price auction, only for the case of $r_1 = r_2 = \frac{1}{2}$. While we can calculate the bidding strategies in more general cases (and they will give us more cumbersome expressions), it turns out that this is the only sub-game that is needed for the analysis of the game; i.e., as we prove later, the publisher always sets $r_1 = r_2 = \frac{1}{2}$ in equilibrium, and other sub-games are all dominated.

Lemma 3 shows that, unlike in Lemma 1, advertisers' strategies under header bidding in an exchange can directly depend on what happens in the other exchange. In particular, when an exchange uses a first-price auction, advertisers within that exchange take the existence of the other exchange, and the number of advertisers within the other exchange, into account when calculating their bids. Interestingly, if an exchange uses a second-price auction, advertisers within that exchange do not take the existence of the other exchange into account, as bidding truthfully continues to be a weakly dominant strategy. This is illustrated in Figure 1 where the gray line depicting the advertisers' bids in a second-price auction is not affected by the number of advertisers in the other exchange, n_2, whereas the advertisers' bids in a first-price auction increase as n_2 increases, i.e., the "outside competition" becomes stronger.



(a) Exchange 2 uses a second-price auction (b) Exchange 2 uses a first-price auction

Figure 1: Bids of advertisers in Exchange 1 as a function of their valuation, for $n_1 = 2$, reserve prices $r_1 = r_2 = \frac{1}{2}$, and different values of n_2. The gray line (i.e., truthful bid) is for when Exchange 1 uses a second-price auction (for any value of n_2); the black curves are for when it uses a first-price auction.

Figure 1 shows that advertisers submit lower bids in a first-price auction than in a second-price auction. However, the lower bids do not imply a lower revenue for the exchange because advertisers pay what they bid, instead of the next highest bid, in a first-price auction. In fact, from the revenue equivalence principle (e.g., see Krishna, 2009) we know that the expected clearing price of a second-price auction is the same as the expected clearing price of a first-price auction when $n_2 = 0$. In other words, the truthful bidding function (i.e., the solid gray lines in Figure 1) in a second-price auction has the same expected clearing price as the solid black curve bidding function (i.e, cases of $n_2 = 0$) in a first-price auction. Therefore, it is easy to see that when $n_2 = 2$, the equilibrium bidding function, represented by the dotted black line, leads to a higher expected clearing price than a first-price auction with $n_2 = 0$, thus also a higher expected clearing price than a second-price auction (for any n_2).

Next, we discuss two important effects of first-price auctions under header bidding:

1. **Exposure to outside competition:** Under header bidding, when Exchange i uses a first-price auction, unlike in Lemma 1, advertisers take the existence of the other exchange into account when calculating their bids. This is because advertisers know that, under header bidding, just being the highest bidder in their own exchange is not sufficient for winning the impression. Note that this aggressive bidding behavior

does not happen if Exchange i uses a second-price auction, where bidding truthfully continues to be a weakly dominant strategy. In other words, by using a first-price auction, an exchange can "expose its advertisers to the outside competition," and, therefore, induce them to bid more aggressively. The exchange benefits from this exposure as it increases its expected clearing price as well as the probability of winning the impression. Note that first-price auctions allow exchanges to expose advertisers to outside competition only under header bidding. In particular, the exposure effect happens because of the *parallel* nature of header bidding, i.e., all advertisers, regardless of what exchange they are in, are competing for the impression simultaneously.

2. **Unified First-price Auction:** If *both* exchanges use first-price auctions with the same reserve price, advertisers' equilibrium bids are as if they are all in one unified first-price auction. In other words, the negative effect of the advertisers being in two separate markets (on the publisher's revenue) disappears.

Next, we analyze the exchanges' choice of first-price versus second-price auctions.

Proposition 4. *Under header bidding, when $r_1 = r_2 = \frac{1}{2}$ and for any values of $n_1, n_2 > 0$, there is a unique equilibrium where both exchanges use first-price auctions.*

Intuitively, the result of Proposition 4 is driven by the outside-competition-exposure effect of first-price auctions under header bidding. In particular, we know from the revenue equivalence principle that if Exchange j did not exist, the revenue of Exchange i under first-price auction would have been the same as under second-price auction. In the presence of Exchange j, the bids in a second-price auction in Exchange i remain truthful, and thus the expected clearing price of Exchange i remains unchanged. However, the outside-competition-exposure effect of first-price auctions, as discussed earlier, induces the advertisers to bid more aggressively (compared to when Exchange j did not exist). As such, by switching to the first-price format, an exchange can increase its expected clearing price, and therefore, its expected revenue. This leads to the unique equilibrium of Proposition 4 where both exchanges use first-price auctions.

The result of Proposition 4 is conditioned on the publisher setting the reserve price $r_1 = r_2 = \frac{1}{2}$ for both exchanges. Next, we show that $r_1 = r_2 = \frac{1}{2}$ is indeed the optimal pair of reserve prices for the publisher. As discussed earlier, we know when both exchanges use first-price auctions with the same reserve prices, advertisers' bids from both exchanges are as if they are all in one unified first-price auction. In other words, the equilibrium in Proposition 4 is equivalent to the equilibrium of a first-price auction with all $n_1 + n_2$ advertisers and reserve price $\frac{1}{2}$. Most importantly, the reserve price $\frac{1}{2}$ is already the optimal reserve price of a unified first-price auction[27], i.e., the publisher can achieve the revenue of Myerson's optimal mechanism (Myerson, 1981) by setting $r_1 = r_2 = \frac{1}{2}$. In more detail, suppose M_1 is the mechanism that runs a simple first-price auction among the n advertisers with reserve price $r = \phi^{-1}(0) = \frac{1}{2}$. Consider the family of mechanisms $M_2(r_1, r_2)$ where we split the advertisers into n_1 and n_2 among the two exchanges (for any choice of n_1 and n_2 summing to n), let the two exchanges decide if they want to use a first-price or a second-price auction for their own group of advertisers with reserve prices r_1 and r_2 respectively, take the two resulting clearing prices from each group, and pick the largest one to be the winner. Myerson's result implies that

$$\text{Revenue}(M_1) \geq \max_{r_1, r_2} \left\{ \text{Revenue}(M_2(r_1, r_2)) \right\}.$$

Now let M_3 be the specific mechanism in the family $M_2(r_1, r_2)$, where both of the reserve prices are equal to $\phi^{-1}(0) = \frac{1}{2}$. By our earlier observation on unified first-price auctions, since both exchanges are using a first-price auction and both reserve prices are equal to $\phi^{-1}(0)$ in M_3, it is essentially identical to a unified first-price auction, i.e. the mechanism

[27] Myerson's optimal reserve price, under both second-price and first-price auctions, when the CDF and PDF of bidders' valuation are F and f, respectively, is $\phi^{-1}(0)$ where $\phi(x) = x - \frac{1-F(x)}{f(x)}$ is Myerson's "virtual valuation." When F is uniform $[0, 1]$, the optimal reserve price becomes $\frac{1}{2}$.

M_1. Moreover, under M_3, from Proposition 4 we know that in the subgame equilibrium between the two exchanges, both the exchanges will choose a first-price auction. Thus we get that

$$\text{Revenue}(M_3) = \text{Revenue}(M_1) \geq \max_{r_1, r_2} \left\{ \text{Revenue}(M_2(r_1, r_2)) \right\}.$$

In particular, this shows that for all splits (n_1, n_2), the revenue of M_3 when setting $r_1 = r_2 = \frac{1}{2}$ is at least as large as that for any pair of reserve prices (including cases when $r_1 \neq r_2$), and this revenue is the maximum possible over all ways of selling the impression. Thus we get the following result.

Corollary 1. *Under header bidding, the publisher can achieve Myerson's optimal revenue.*

To highlight the significance of Corollary 1, note that Myerson's setting has far fewer constraints for the seller than our setting. In particular, under Myerson's setting, the seller has direct access to the bidders, and full control over the selling mechanism. In our setting, there are intermediaries (i.e., exchanges) that choose mechanisms that optimize their own revenue. Yet, we get a unique equilibrium where the intermediaries' strategies optimize the seller's revenue. Interestingly, the revenue of the exchanges is not necessarily optimized in this equilibrium, e.g., when $n_1 = 3$ and $n_2 = 1$, Exchange 1 is better off when both exchanges use second-price auctions than when they both use first-price.

Intuitively, each exchange deviates to a first-price auction to increase its expected clearing price, and, therefore, increase its expected seller-side revenue. However, when both exchanges move to first-price auctions, each faces a higher expected clearing price from the other exchange. Therefore, an exchange may end up with a lower equilibrium revenue when both exchanges move to first-price auctions, than when they both use second-price auctions. This resembles (even though it is not mathematically equivalent to) a prisoner's dilemma situation for exchanges with respect to their choices of their auction format. As we show later in Proposition 5, the negative effect of moving to first-price auctions for the exchanges becomes even stronger when we take buyer-side fees into account.

Proposition 4 provides a new explanation for why ad exchanges moved from second-price to first-price auctions. Our explanation is consistent with the timing of the transition to first-price auctions as if it were triggered by the publisher's adoption of header bidding and is consistent with how this market evolved over time. The market share of header bidding among the top publishers in the US grew from 0% to over 70% from 2014 to 2016.[28] Exchanges started experimenting with first-price auctions in 2017 for the first time, and all major exchanges fully moved to first-price auction by early 2019.[29]

A disparity in how major platforms like Google and Facebook sell display advertising impressions is also consistent with our explanation. Google only uses first-price auctions in its exchange platform, where it is an intermediary. For selling its own inventory such as impressions on YouTube, however, since header bidding is not used (i.e., advertisers have to purchase the impressions directly from Google), Google still uses second-price auctions.[30] Similarly, Facebook uses a generalized version of second-price auctions to sell its display advertising impressions directly to advertisers. The fact that these large firms have adopted first-price auctions only in situations where they act as intermediaries provides additional support for our explanation.

Next, we continue our analysis of the game. Note that, since the publisher can achieve the optimal revenue by setting $r_1 = r_2 = \frac{1}{2}$, we do not have to solve for other sub-games with other values of r_i. Thus, we move to advertisers' equilibrium strategies regarding which exchange to join, and exchanges' equilibrium fees under header bidding. The following proposition summarizes the exchanges' equilibrium fees, and advertisers' choice of exchanges under header bidding.

Proposition 5. *Under header bidding, both exchanges set their buyer-side fees to zero, i.e., $f_1 = f_2 = 0$, in equilibrium. The advertisers are indifferent about which exchange to join.*

[28] https://www.businessinsider.com/header-bidding-gains-momentum-drives-up-publisher-ad-revenue-2016-5

[29] https://adexchanger.com/platforms/big-changes-coming-auctions-exchanges-roll-dice-first-price/ and https://adage.com/article/digital/google-adx-moving-a-price-auction/316894

[30] https://adexchanger.com/online-advertising/google-switches-to-first-price-auction/

From Proposition 4, we know that both exchanges use first-price auctions in equilibrium. This reduces the market with two exchanges to a unified first-price auction. Since advertisers are forward-looking, they know that the choice of exchange does not affect their probability of winning or their expected utility. In other words, regardless of the choice of the exchange, an advertiser wins if and only if he has the highest bid among all advertisers across all exchanges. Therefore, for any buyer-side fees f_1 and f_2, an advertiser's optimal strategy is to choose the exchange with the lower fee. Given the advertisers' strategies, the exchanges set their buyer-side fees to zero in equilibrium. The finding of Proposition 5 is in line with industry reports that show, after adoption of first-price auctions, many exchanges have reduced, or completely dropped, buyer-side fees.[31]

Note that, using Proposition 2, we know exchanges can extract positive equilibrium revenue through buyer-side fees when the publisher uses waterfalling. However, Proposition 5 shows that, when the publisher uses header bidding, the exchange's ability to charge buyer-side fees disappears. Intuitively, this is because the exchanges can differentiate in terms of their position in the waterfall sequence and the number of advertisers they have, under waterfalling. For example, if an exchange has more advertisers on its platform, or if it has a more favorable position in the waterfall sequence, it can charge higher buyer-side fees in equilibrium. On the other hand, when the publisher uses header bidding and the exchanges use first price auctions, the exchanges' ability to differentiate themselves using their position in the sales channel disappears. Advertisers choose the exchange with the lowest fee, and the exchanges' profit from buyer-side fees declines to zero.

Finally, we should mention that while the exchanges' buyer-side fees decline to zero in equilibrium, exchanges still obtain positive revenue through seller-side fees. In other words, the exchanges still benefit from attracting more advertisers because by having more advertisers, they can increase their seller-side revenue. We should also note that, in our model, the exchanges' buyer-side fees decline all the way to zero because, to facilitate exposition,

[31] https://adexchanger.com/platforms/rubicon-project-eliminates-buy-side-fees/

we have assumed that the exchanges are ex-ante identical (no differentiation). In practice, exchanges can horizontally differentiate though the tools and services that they offer to the advertisers. Indeed, in Section B.1.2, we consider an extension of our main model with horizontally differentiated exchanges, and show that the exchanges obtain positive buyer-side revenue in equilibrium when they are horizontally differentiated.

4 Conclusion

In this paper, we propose a simple model of real-time bidding in display advertising to analyze the evolution of selling mechanisms in this market, and the consequences for advertisers, publishers and exchange platforms. We show that, when exchanges were using second-price auctions and advertisers' exchange affiliations were unchanged, the publisher's revenue when using header bidding is always higher than waterfalling; this result explains the rapid adoption of header bidding by publishers in recent years.

Our results also provide a new explanation for why exchange platforms moved from second-price auctions to first-price. Second-price auctions have been the industry standard in online advertising for over a decade. In fact, they are still the dominant selling mechanism in search advertising as well as display advertising markets in which publishers directly sell to advertisers (without going through a third-party exchange). For example, Google uses second-price auctions for selling YouTube and AdSense impressions, and Facebook uses a generalized form of second-price auctions to sell its display advertising inventory. We argue in this paper that this move may be due to the wide adoption of header bidding. Our results provide managerial implications for advertisers, publishers, and exchanges in the online advertising industry.

Implications for Advertisers. In the past few years, the selling mechanism in real-time bidding market has dramatically changed: first, publishers moved from waterfalling to header bidding, and then exchanges moved from second-price to first-price auctions. This leaves the advertisers with much uncertainty on how to adjust their bidding strategies under the new mechanism. Our results show that advertisers should shade their bids using the same methods as in a standard first-price auction. The degree of shading depends on the number of other advertisers in the market as well as their distribution of values for the impression. Under header bidding, in contrast with waterfalling, advertisers should consider every other advertiser in the market as competition, not only those who use the same exchange.

Another important implication for advertisers is regarding the choice of exchange. Previously, under waterfalling, advertisers had to pay attention to the position of an exchange in the sequence of the waterfall, as well as the number of other advertisers in each exchange. Using the exchange with the lowest fee was not necessarily the optimal strategy. This continues to hold under header bidding with second-price auctions. However, under header bidding with first-price auctions, submitting the bid through different exchanges does not affect the final price and allocation of an advertiser; as such, the optimal strategy of an advertiser is the simple one of using the exchange with the lowest fee.

Implications for Exchanges. Under waterfalling, an exchange could use its position in the waterfall sequence to differentiate itself from other exchanges. When exchanges use second-price auctions, they can use the set of the advertisers that they have to differentiate from other exchanges; intuitively, an advertiser benefits from being in an exchange where other advertisers with similar valuations are part of. However, the combination of header bidding and first-price auctions put exchanges in direct competition. While the move to first-price auction was necessary for an exchange to survive in the short run after the publishers adopted header bidding, after taking its effect on advertisers' choices of exchanges into account, our results show that the move will lower the exchanges' equilibrium buyer-side fees in the long-run. This is consistent with several industry reports indicating a steep decline in exchange fees since the adoption of first-price auctions.[32]

In order to avoid head-on competition, exchanges can no longer rely on their position

[32] https://adexchanger.com/platforms/big-changes-coming-auctions-exchanges-roll-dice-first-price/

within the selling mechanism as a point of differentiation; they have to create new strategies to differentiate in this market. For example, they can leverage their information about transactions in this market and offer analytical tools to advertisers who use their platform. Alternatively, they can vertically differentiate by filtering out low-quality (or suspicious/fraudulent) impressions to guarantee certain viewability rates.

Implications for Publishers. Our results indicate that publishers mainly benefit from the adoption of first-price auctions by exchanges. The direct benefit of first-price auctions for the publishers is that all advertisers directly compete with each other when all exchanges use first-price auctions. In other words, even though the advertisers only compete within an exchange, they take the existence of other exchanges (and other advertisers in those exchanges) into account when optimizing their bids. This effectively becomes equivalent to as if all advertisers were in the same unified first-price auction. In other words, first-price auctions under header bidding eliminate the negative effect of advertisers being separated into multiple exchanges on the publisher's revenue. There is already some early evidence of this improved revenue for publishers and a more competitive market for advertisers as a result of the move to first-price auctions in the industry.[33]

We also show that first-price auctions under header bidding allow publishers to achieve the revenue of Myerson's optimal mechanism (Myerson, 1981). In other words, just by setting the reserve prices optimally, the publisher can achieve the revenue of the optimal mechanism that has direct access to advertisers (i.e., without the interference of intermediaries) and can use any (individually rational) pricing and allocation.

Finally, from a computational point of view, the move to first-price auction simplifies the publisher's choice of optimal reserve prices. Under waterfalling, and also when exchanges use second-price auctions, the publisher has to solve a joint optimization problem and set asymmetric reserve prices even when the exchanges and the advertisers are symmetric. The reserve price of an exchange had to take the expected revenue from other exchanges into

[33] https://www.blog.google/products/admanager/rolling-out-first-price-auctions-google-ad-manager-partners/

account. These interactions made the calculation of the optimal reserve prices extremely complicated. However, under header bidding with first-price auctions, the optimal reserve price of each exchange is independent of all other exchanges, and happens to be the standard optimal reserve price of Myerson (1981).

Limitations and Future Research. Our results shed light on the evolution of the selling mechanism in the RTB market over time, and provide insights for managers in this industry on how to buy and sell display advertising inventory in the current market. In our model, we make several simplifying assumptions. First and foremost, we assume the advertisers' valuations are independent draws from the same distribution. In reality, advertisers may be asymmetric in their distributions of valuations; furthermore, their valuations may be correlated. Intuitively, Zeithammer (2019) shows that asymmetric distributions can favor adoption of first-price auctions, whereas Milgrom and Weber (1982) show that correlation in advertisers' valuations favors adoption of second-price auctions. The question of the optimal mechanism design in display advertising markets with general distributions is beyond the scope of this paper, and thus left to future research.

Similarly, our model assumes that the number of advertisers in the market is common knowledge. While this might hold in mature markets where advertisers have been competing with each other for a long time, it does not hold in other situations. In particular, an advertiser may not know how many other advertisers may be interested in a given impression. This assumption significantly simplifies our analysis and allows us to use the framework in Myerson (1981). Studying advertising auctions where the number of competing advertisers is not known is an interesting research direction that can be explored in future research.

Our work is among a small, but growing, set of papers that study the fast growing RTB market. While we focused on the role of exchanges in this paper, future research can explore other entities in this market such as demand-side platforms (DSP), supply-side platforms (SSP), and data-management platforms (DMP). In particular, since competing advertisers sometimes use the same DSP, and competing publishers can use the same SSP, extending our analysis of intermediaries to such buyer and seller agents can lead to further interesting economic insights. Another direction to explore is how privacy regulation affects the role of DMPs, and the optimal level of information sharing in this market.

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A Appendix

A.1 Analyses and Proofs

Proof of Lemma 1. If Exchange i is using a second-price auction, it is easy to verify that bidding truthfully is a weakly dominant strategy for the advertisers.

Suppose now that Exchange i is using a first-price auction, and let $b_i(v)$ be a symmetric increasing bidding function of advertisers in Exchange i. The function $b_i(v)$ should satisfy the boundary condition $b_i(r_i) = r_i$. The expected utility of an advertiser with valuation $v \geq r_i$ if he bids $b_i(x)$ instead of $b_i(v)$, for some $x \geq r_i$, is

$$u(x) = F(x)^{n_i-1}(v - b_i(x)) = x^{n_i-1}(v - b_i(x)).$$

To have an equilibrium, $u(x)$ must be maximized for $x = v$. We have $\frac{\partial u}{\partial x} = (n_i - 1)x^{n_i-2}(v - b_i(x)) - x^{n_i-1}b_i'(x)$. Since $u(x)$ is maximized for $x = v$, we get $\frac{\partial u}{\partial x}\Big|_{x=v} = 0$, which gives the differential equation

$$b_i'(v) = \frac{(n_i - 1)(v - b_i(v))}{v}.$$

The solution to this differential equation is $b_i(v) = v - \frac{v}{n_i} + \frac{C}{v^{n_i-1}}$, where C is a constant. Using the boundary condition $b_i(r_i) = r_i$, we can find that $C = \frac{r_i^{n_i}}{n_i}$. Therefore, in equilibrium, an advertiser in Exchange i with valuation $v \geq r_i$ bids $b_i(v) = v - \frac{v}{n_i} + \frac{r_i^{n_i}}{n_i \cdot v^{n_i-1}}$. □

Proof of Lemma 2. From Myerson (1981), we know that the virtual valuations in Exchange 2 are defined as $\phi(x) = x - \frac{1-F(x)}{F'(x)} = 2x - 1$. Therefore, the optimal reserve price for Exchange 2, regardless of the auction format (according to Myerson (1981)), is $r_2 = \phi^{-1}(0) = \frac{1}{2}$. Consequently, the expected revenue of the publisher from Exchange 2 is

$$\begin{aligned} w_2 &= (1-f) \cdot \left[n_2 \left(r_2(1 - F(r_2))F(r_2)^{n_2-1} + \int_{r_2}^{1} y(1 - F(y))(n_2 - 1)F(y)^{n_2-2}F'(y)\,dy\right)\right] \\ &= (1-f) \cdot \left[1 - \frac{2}{n_2+1} + \frac{1}{(n_2+1) \cdot 2^{n_2}}\right]. \end{aligned}$$

The expected revenue of the publisher from Exchange 2 is the publisher's expected revenue from not selling the impression in Exchange 1; i.e., when analyzing Exchange 1, w_2 is the seller's value for keeping the item. Therefore, we know from Myerson (1981) that the optimal reserve price for Exchange 1 is $r_1 = \phi^{-1}\left(\frac{w_2}{1-f}\right) = 1 - \frac{1}{n_2+1} + \frac{1}{(n_2+1)\cdot 2^{n_2+1}}$. Note that, using Myerson (1981), this is the optimal reserve price for both first-price and second-price auction formats. □

Proof of Proposition 1. This result comes from the *revenue equivalence principle*. Suppose that Exchange i has n_i advertisers and it is using a second-price auction with reserve price r_i. Then its expected clearing price is

$$p_i^{SP} = n_i \left(r_i(1 - F(r_i))F(r_i)^{n_i-1} + \int_{r_i}^{1} y(1 - F(y))(n_i - 1)F(y)^{n_i-2}F'(y)\, dy \right).$$

Similarly, suppose that Exchange i has n_i advertisers and it is using a first-price auction with reserve price r_i. Let $b_i(v)$ be the bidding function of its advertisers. Then its expected clearing price is

$$p_i^{FP} = n_i \int_{r_i}^{1} b_i(y)F(y)^{n_i-1}F'(y)\, dy.$$

It is easy to verify that for $b_i(v) = v - \frac{v}{n_i} + \frac{r_i^{n_i}}{n_i \cdot v^{n_i-1}}$, which is the bidding function of advertisers in a first-price auction (Lemma 1), it holds that $p_i^{SP} = p_i^{FP}$. Therefore, for a fixed reserve price, exchanges are indifferent between running a first-price auction or a second-price auction, as both will give the same expected revenue. When the publisher sets the reserve prices, the optimal reserve prices r_1 and r_2 are the same in a first-price auction as in a second-price auction (by Lemma 2). Therefore, the publisher's and the exchanges' expected revenues are the same in a first-price auction as in a second-price auction.

Finally, an advertiser's winning probability and expected payment are the same in a first-price and in a second-price auction. The winning probability is the probability that the advertiser has the highest valuation among the advertisers in his exchange, and the expected payments are simply $\frac{p_i^{SP}}{n_i}$ and $\frac{p_i^{FP}}{n_i}$, for a second-price and a first-price auction respectively, which are equal. Therefore advertisers' utilities are not affected either, by the auction format. □

Proof of Proposition 2. If Exchange 1 has n_1 advertisers and Exchange 2 has n_2 advertisers, then the expected utility of an advertiser in Exchange 1 (ignoring the fee for now) is

$$\begin{aligned} u_1(n_1, n_2) = &\int_{r_1}^{1} F(y)^{n_1-1} y F'(y)\, dy - \\ &\left(r_1(1 - F(r_1))F(r_1)^{n_1-1} + \int_{r_1}^{1} y(1 - F(y))(n_1 - 1)F(y)^{n_1-2}F'(y)\, dy\right) \\ = &\frac{1}{n_1(n_1+1)} - \frac{r_1^{n_1}}{n_1} + \frac{r_1^{n_1+1}}{n_1+1}. \end{aligned} \tag{1}$$

The expected utility of an advertiser in Exchange 2 (again ignoring the fee) is

$$\begin{aligned} u_2(n_1, n_2) = &F(r_1)^{n_1} \cdot \left[\int_{r_2}^{1} F(y)^{n_2-1} y F'(y)\, dy - \right. \\ &\left.\left(r_2(1 - F(r_2))F(r_2)^{n_2-1} + \int_{r_2}^{1} y(1 - F(y))(n_2 - 1)F(y)^{n_2-2}F'(y)\, dy\right)\right] \\ = &r_1^{n_1} \cdot \left(\frac{1}{n_2(n_2+1)} - \frac{r_2^{n_2}}{n_2} + \frac{r_2^{n_2+1}}{n_2+1}\right). \end{aligned} \tag{2}$$

Note that in the expressions above, r_1 and r_2 are the optimal reserve prices from Lemma 2, i.e. r_1 is a function of n_2 and $r_2 = \frac{1}{2}$.

Consider the case where $n_1 = 0$, $n_2 = n$, and $f_1 = 0$, i.e. all advertisers are in Exchange 2 while Exchange 1 has a zero fee. Then Exchange 2 can charge a positive fee $f_2 > 0$ such that no advertiser benefits by moving to Exchange 1 (even though Exchange 1's fee is 0). More specifically, the maximum fee Exchange 2 can charge so that none of its advertisers wants to move is

$$f_2^* = u_2(0, n) - u_1(1, n-1) = \frac{n \cdot (4^n - 1) - (2^n - 1)^2 - n^2 \cdot 2^n}{n^2(n+1) \cdot 2^{2n+1}} > 0.$$

In other words, there is an equilibrium where the total exchange buyer-side revenue is positive, i.e. $n_1 f_1 + n_2 f_2 > 0$. □

Proof of Proposition 3. Recall that we are analyzing the situation where the number of advertisers in both exchanges is fixed and nonzero and both exchanges are using second-price auctions. First, note that it is weakly dominant for advertisers to bid truthfully under both waterfalling and header bidding. Therefore, moving to header bidding does not affect advertisers' bidding strategies. Moreover, for any pair of reserve prices r_1 and r_2, the publisher's revenue under header bidding is greater than or equal to that under waterfalling. This is because, under header bidding, the publisher can see the clearing prices of both exchanges before deciding which exchange wins the impression, whereas under waterfalling, the publisher has to accept or decline the clearing price of Exchange 1 before seeing the clearing price of Exchange 2. Therefore, assuming that r_1^* and r_2^* are the optimal reserve prices under waterfalling, we know that the publisher's revenue when using r_1^* and r_2^* under header bidding is greater than when using r_1^* and r_2^* under waterfalling. As such, the publisher's optimal revenue under header bidding is greater than the publisher's optimal revenue under waterfalling. □

Proof of Lemma 3. If an exchange is using a second-price auction, then it is a weakly dominant strategy for its advertisers to bid their true valuation.

If both exchanges use first-price auctions with the same reserve price, then for the advertisers this is equivalent to a global first-price auction. Therefore, their bidding function is the standard bidding function for a first-price auction with $n_1 + n_2$ advertisers.

It remains to consider the case where one of the exchanges uses a first-price auction and the other uses a second-price auction. W.l.o.g. suppose that Exchange 1 is using a first-price auction and Exchange 2 is using a second-price auction.

Consider a symmetric equilibrium bidding function $b(v)$ for the advertisers in Exchange 1. Let $r = \frac{1}{2}$. The expected utility of an advertiser with valuation $v \geq r$ if his bid is $b(x)$ instead of $b(v)$ is

$$u(x) = (n_2 F(b(x))^{n_2-1} - (n_2-1)F(b(x))^{n_2})F(x)^{n_1-1}(v - b(x)). \tag{3}$$

To have an equilibrium, this function must be maximized for $x = v$.

We start with the simple case where $n_2 \leq 1$ and $n_1 \geq 2$. The utility function becomes

$$u(x) = F(x)^{n_1-1}(v - b(x)).$$

Therefore, advertisers in Exchange 1 can ignore Exchange 2 and bid as if they are in a simple first-price auction with n_1 advertisers. In this case, the bidding function is

$$b(v) = \begin{cases} 0 & \text{, if } v < r, \\ v - \frac{v}{n_1} + \frac{r^{n_1}}{n_1 \cdot v^{n_1-1}} & \text{, if } v \geq r. \end{cases}$$

Next, consider the case where $n_1 = 1$ and $n_2 \geq 2$. The utility function becomes

$$u(x) = (n_2 F(b(x))^{n_2-1} - (n_2-1)F(b(x))^{n_2})(v - b(x)).$$

Let $z(v)$ be the point y that maximizes the function $u(b^{-1}(y))$. For v that satisfy $z(v) \geq r$, we get $b(v) = z(v)$. For v that satisfy $z(v) < r$, we have $b(v) = r + \epsilon$, where $0 < \epsilon < v - r$.[34] Let t be such that $z(t) = r$. Then the bidding function is

$$b(v) = \begin{cases} 0 & \text{, if } v \leq r, \\ r + \epsilon & \text{, if } r < v < t, \\ z(v) & \text{, if } v \geq t. \end{cases}$$

[34]The role of ϵ here is to break the tie between the two exchanges in the case there is only one advertiser in Exchange 2 who bids above the reserve price. If we change the tie-breaking rule to say that the exchange with the first-price auction wins in case of a tie, then we can remove ϵ from the equations.

From this, we can obtain the following special cases.

- For $n_1 = 1$ and $n_2 = 2$, it is $b(v) = \begin{cases} 0 & , \text{if } v \leq r, \\ r + \epsilon & , \text{if } v > r. \end{cases}$

- For $n_1 = 1$ and $n_2 = 3$, it is $b(v) = \begin{cases} 0 & , \text{if } v \leq r, \\ r + \epsilon & , \text{if } r < v < \frac{5}{6}, \\ \frac{1}{16}\left(6v + 9 - \sqrt{36v^2 - 84v + 81}\right) & , \text{if } v \geq \frac{5}{6}. \end{cases}$

Some other trivial cases are the following.

- For $n_1 = 1$ and $n_2 = 1$, it is $b(v) = \begin{cases} 0 & , \text{if } v \leq r, \\ r + \epsilon & , \text{if } v > r. \end{cases}$

- For $n_1 = 1$ and $n_2 = 0$, it is $b(v) = \begin{cases} 0 & , \text{if } v < r, \\ r & , \text{if } v \geq r. \end{cases}$

Finally, let's consider the case with $n_1 = 2$ and $n_2 = 2$. To find b, we need to solve the differential equation $\frac{\partial u}{\partial x}\Big|_{x=v} = 0$ with boundary condition $b(r) = r$. The differential equation is

$$b'(v) = \frac{(2 - b(v))b(v)(v - b(v))}{v\left(-3b(v)^2 + 2(v + 2)b(v) - 2v\right)}, \tag{4}$$

and its solution is plotted in Figure 2. Note that, even without a closed-form solution for $b(v)$ for the case of $n_1 = n_2 = 2$, we can prove Proposition 4 analytically by using an analytical lower bound for $b(v)$. □

Proof of Proposition 4. If both exchanges use a second-price auction, the expected seller-

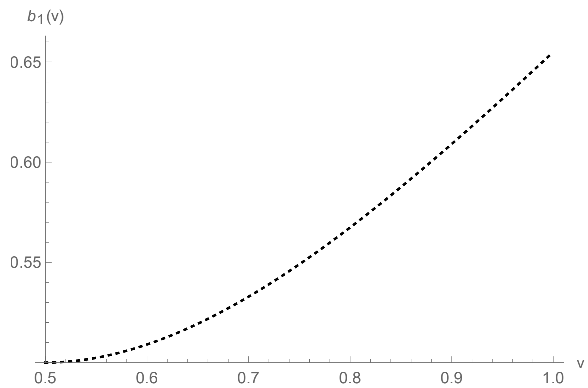


Figure 2: Bidding function of advertisers in Exchange 1 for $n_1 = 2$ and $n_2 = 2$, when Exchange 1 is using a first-price auction and Exchange 2 is using a second-price auction.

side revenue of Exchange 1 is

$$
\begin{aligned}
L_1^{(SP,SP)} =& f \cdot \Big\{ r_1 \cdot n_1(1-F(r_1))F(r_1)^{n_1-1}F(r_2)^{n_2} + \\
& \frac{1}{2} \cdot r_2 \cdot n_1 n_2 (1-F(r_1))F(r_1)^{n_1-1}(1-F(r_2))F(r_2)^{n_2-1} + \\
& F(r_2)^{n_2} \left(\int_{r_1}^{a} y \cdot n_1(n_1-1)(1-F(y))F(y)^{n_1-2}F'(y)\, dy \right) + \\
& n_2(1-F(r_2))F(r_2)^{n_2-1} \left(\int_{r_2}^{a} y \cdot n_1(n_1-1)(1-F(y))F(y)^{n_1-2}F'(y)\, dy + \right) \\
& \int_{r_2}^{a} \left[n_2(n_2-1)(1-F(z))F(z)^{n_2-2}F'(z) \left(\int_{z}^{a} y \cdot n_1(n_1-1)(1-F(y))F(y)^{n_1-2}F'(y)\, dy + \right) \right] dz \Big\},
\end{aligned}
$$

where $a = 1$ is the maximum possible valuation of an advertiser. The expected seller-side revenue of Exchange 2 is

$$
\begin{aligned}
L_2^{(SP,SP)} =& f \cdot \Big\{ r_2 \cdot F(r_1)^{n_1} n_2(1-F(r_2))F(r_2)^{n_2-1} + \\
& \frac{1}{2} \cdot r_2 \cdot n_1 n_2 (1-F(r_1))F(r_1)^{n_1-1}(1-F(r_2))F(r_2)^{n_2-1} + \\
& F(r_1)^{n_1} \left(\int_{r_2}^{a} y \cdot n_2(n_2-1)(1-F(y))F(y)^{n_2-2}F'(y)\, dy \right) + \\
& n_1(1-F(r_1))F(r_1)^{n_1-1} \left(\int_{r_2}^{a} y \cdot n_2(n_2-1)(1-F(y))F(y)^{n_2-2}F'(y)\, dy \right) + \\
& \int_{r_2}^{a} \left[n_2(n_2-1)(1-F(z))F(z)^{n_2-2}F'(z) \left(\int_{r_1}^{z} z \cdot n_1(n_1-1)(1-F(y))F(y)^{n_1-2}F'(y)\, dy \right) \right] dz \Big\}.
\end{aligned}
$$

If both exchanges use a first-price auction, the expected seller-side revenue of Exchange 1 is

$$\begin{aligned}L_1^{(FP,FP)} =& f \cdot \Big\{ r_1 n_1 F(r_1)^{n_1-1}(1-F(r_1))F(r_2)^{n_2} + \\ & \int_{r_1}^{a} y(n_1+n_2-1)n_1(1-F(y))F(y)^{n_1+n_2-2}F'(y)\,dy \Big\}.\end{aligned}$$

The expected seller-side revenue of Exchange 2 is

$$\begin{aligned}L_2^{(FP,FP)} =& f \cdot \Big\{ r_2 n_2 F(r_2)^{n_2-1}(1-F(r_2))F(r_1)^{n_1} + \\ & \int_{r_2}^{a} y(n_1+n_2-1)n_2(1-F(y))F(y)^{n_1+n_2-2}F'(y)\,dy \Big\}.\end{aligned}$$

Now suppose that Exchange 1 is using a first-price auction, while Exchange 2 is using a second-price auction. Let $b_1(v)$ be the bidding function of an advertiser in Exchange 1.[35] Then, the expected seller-side revenue of Exchange 1 is

$$\begin{aligned}L_1^{(FP,SP)}(b_1) =& f \cdot \Big\{ F(r)^{n_2} \int_r^a b_1(x) n_1 F(x)^{n_1-1} F'(x)\,dx + \\ & n_2 F(r)^{n_2-1}(1-F(r)) \int_r^a b_1(x) n_1 F(x)^{n_1-1} F'(x)\,dx + \\ & \int_r^a \left[b_1(z) n_1 F(z)^{n_1-1} F'(z) \left(\int_r^{b_1(z)} (n_2-1)n_2(1-F(x))F(x)^{n_2-2}F'(x)\,dx \right) \right] dz \Big\},\end{aligned}$$

where $r = \frac{1}{2}$ is the common reserve price. The expected seller-side revenue of Exchange 2 is

$$\begin{aligned}L_2^{(FP,SP)}(b_1) =& f \cdot \Big\{ r \cdot n_2 F(r)^{n_2-1}(1-F(r))F(r)^{n_1} + \\ & F(r)^{n_1} \int_r^a x(n_2-1)n_2(1-F(x))F(x)^{n_2-2}F'(x)\,dx + \\ & \int_r^a n_1 F(z)^{n_1-1} F'(z) \left(\int_{b_1(z)}^a x(n_2-1)n_2(1-F(x))F(x)^{n_2-2}F'(x)\,dx \right) dz \Big\}.\end{aligned}$$

Finally suppose that Exchange 2 is using a first-price auction, while Exchange 1 is using a second-price auction. Let $b_2(v)$ be the bidding function of an advertiser in Exchange 2. Then, the expected seller-side revenue of Exchange 1 is

$$\begin{aligned}L_1^{(SP,FP)}(b_2) =& f \cdot \Big\{ r \cdot n_1 F(r)^{n_1-1}(1-F(r))F(r)^{n_2} + \\ & F(r)^{n_2} \int_r^a x(n_1-1)n_1(1-F(x))F(x)^{n_1-2}F'(x)\,dx +\end{aligned}$$

[35] Here we assume a symmetric equilibrium bidding strategy for advertisers in Exchange 1.

$$\int_r^a n_2 F(z)^{n_2-1} F'(z) \left(\int_{b_2(z)}^a x(n_1-1)n_1(1-F(x))F(x)^{n_1-2}F'(x)\,dx \right) dz \Bigg\}.$$

The expected seller-side revenue of Exchange 2 is

$$\begin{aligned} L_2^{(SP,FP)}(b_2) =& f \cdot \Bigg\{ F(r)^{n_1} \int_r^a b_2(x) n_2 F(x)^{n_2-1} F'(x)\,dx + \\ & n_1 F(r)^{n_1-1}(1-F(r)) \int_r^a b_2(x) n_2 F(x)^{n_2-1} F'(x)\,dx + \\ & \int_r^a \left[b_2(z) n_2 F(z)^{n_2-1} F'(z) \left(\int_r^{b_2(z)} (n_1-1)n_1(1-F(x))F(x)^{n_1-2}F'(x)\,dx \right) \right] dz \Bigg\}. \end{aligned}$$

Consider the following payoff matrix, $M_{n_1,n_2}(b_1,b_2)$, of the game between the exchanges.

		Exchange 2	
		SP	FP
Exchange 1	SP	$\left\{L_1^{(SP,SP)}, L_2^{(SP,SP)}\right\}$	$\left\{L_1^{(SP,FP)}(b_2), L_2^{(SP,FP)}(b_2)\right\}$
	FP	$\left\{L_1^{(FP,SP)}(b_1), L_2^{(FP,SP)}(b_1)\right\}$	$\left\{L_1^{(FP,FP)}, L_2^{(FP,FP)}\right\}$

To show that (FP,FP) is the unique equilibrium, it is sufficient to show that

$$L_1^{(FP,FP)} > L_1^{(SP,FP)}(b_2) \text{ and } L_2^{(FP,FP)} > L_2^{(FP,SP)}(b_1) \text{ and } L_1^{(FP,SP)}(b_1) > L_1^{(SP,SP)}. \tag{5}$$

Consider a pointwise lower bound $b_1'(v)$ of the bidding function $b_1(v)$. Note that $L_1^{(FP,SP)}(b_1) \geq L_1^{(FP,SP)}(b_1')$ and $L_2^{(FP,SP)}(b_1) \leq L_2^{(FP,SP)}(b_1')$. This is because if the advertisers in Exchange 1 decrease their bids, Exchange 1's revenue will go down (lower chance of winning and lower clearing price) while Exchange 2's revenue will go up (higher chance of winning).

Similarly, if $b_2'(v)$ is a pointwise lower bound of $b_2(v)$, it holds that $L_1^{(SP,FP)}(b_2) \leq L_1^{(SP,FP)}(b_2')$.

Therefore, it is sufficient to show that

$$L_1^{(FP,FP)} > L_1^{(SP,FP)}(b_2') \text{ and } L_2^{(FP,FP)} > L_2^{(FP,SP)}(b_1') \text{ and } L_1^{(FP,SP)}(b_1') > L_1^{(SP,SP)}. \tag{6}$$

for some lower bound functions b_1' and b_2'.

Consider the function

$$b_1'(v) = v - \frac{v}{n_1} + \frac{1}{n_1 \cdot 2^{n_1} \cdot v^{n_1-1}}.$$

This is how advertisers in Exchange 1 would bid if Exchange 1 was running a first-price auction and advertisers were completely ignoring the existence of Exchange 2. When the advertisers consider Exchange 2, their bids in equilirium can only increase, because now they have to compete with a larger outside option. Therefore, b_1' is a pointwise lower bound of b_1.

Similarly, the function $b_2'(v) = v - \frac{v}{n_2} + \frac{1}{n_2 \cdot 2^{n_2} \cdot v^{n_2-1}}$. is a pointwise lower bound of b_2.

For these lower bounds, the inequalities in (6) become simple inequalities that involve only n_1 and n_2. Therefore, for given n_1 and n_2, it is easy to verify them.

We now consider all the cases for $n_2 \geq n_1 > 0$ and $n_1 + n_2 \leq n$.[36]

- For $n_1 = 2$ and $n_2 = 2$, the matrix $M_{2,2}$ for the lower bound bidding functions b_1' and b_2' is

$M_{2,2}(b_1', b_2') = f \cdot$

$\{0.272917, 0.272917\}$	$\{0.208547, 0.334728\}$
$\{0.334728, 0.208547\}$	$\{0.30625, 0.30625\}$

- For $n_1 = 1$ and $n_2 = 3$, it is

$M_{1,3}(b_1', b_2') = f \cdot$

$\{0.078125, 0.484375\}$	$\{0.03125, 0.53125\}$
$\{0.125, 0.4375\}$	$\{0.153125, 0.459375\}$

- For $n_1 = 1$ and $n_2 = 2$, it is

$M_{1,2}(b_1', b_2') = f \cdot$

$\{0.125, 0.354167\}$	$\{0.0625, 0.416667\}$
$\{0.1875, 0.291667\}$	$\{0.177083, 0.354167\}$

- For $n_1 = 1$ and $n_2 = 1$, it is

$M_{1,1}(b_1', b_2') = f \cdot$

$\{0.1875, 0.1875\}$	$\{0.125, 0.25\}$
$\{0.25, 0.125\}$	$\{0.208333, 0.208333\}$

[36]The cases with $n_1 \geq n_2$ are symmetric.

We can see that for all cases, the inequalities in (6) are satisfied. Therefore, for all cases, (FP, FP) is the unique equilibrium. □

Proof of Proposition 5. Since advertisers are forward looking, at the time of choosing between the exchanges, they know that the reserve prices will be set at $r_1 = r_2 = \frac{1}{2}$ and both exchanges will use first-price auctions. Intuitively, this implies that the exchanges are in a Bertrand competition when setting their buyer-side fees to attract advertisers. In the following, we formalize this intuition.

If $f_1 \neq f_2$, it is optimal for an advertiser to choose the exchange with the lower fee. Therefore, in any pure-strategy Nash equilibrium of the game, we must have $f_1 = f_2$, otherwise, the exchange with the lower fee benefits from increasing its fee to the fee of the other exchange minus ϵ (where ϵ is a sufficiently small positive real number) and still get all of the advertisers. Finally, it is easy to see that $f_1 = f_2 = 0$ is the only equilibrium of the game. If the fees are larger than zero, i.e., $f_1 = f_2 > 0$, at least one exchange benefits from lowering its fee by ϵ to get all of the advertisers. □

A.2 Alternative Timelines

A.2.1 Exchanges decide the auction format before the publisher sets the reserve prices

In this section we explore an alternative timeline of the game. More specifically, we focus on step 3 (where the publisher sets the reserve prices) and step 4 (where the exchanges decide the auction formats) of the main model (see Section 2.1 for the main timeline). We show that Proposition 6 below (the analog of Proposition 4) is robust under a change in the order of these steps.

Proposition 6. *Under header bidding and for any values of $n_1, n_2 > 0$, there is a unique equilibrium where both exchanges use first-price auctions.*

The proof of Proposition 6 is available in Web Appendix C.2.

A.2.2 Advertisers choose an exchange after exchanges decide the auction format

In this section, we continue with the timeline of Section A.2.1 but with an additional change. We assume that the advertisers decide which exchange to join (step 2 in Section 2.1) after the exchanges decide their auction format and before the publisher sets the reserve prices. We show that Proposition 7 below (the analog of Proposition 4) holds in this new timeline.

Proposition 7. *Under header bidding, either both exchanges use first-price auctions or all advertisers join the same exchange in equilibrium.*

The proof of Proposition 7 is available in Web Appendix C.2.

B Horizontally Differentiated Exchanges

B.1 Buyer-Side Fees with Loyals and Switchers

In Proposition 5, we see that under header bidding with first-price auctions, the buyer-side fees decline to 0. If we relax the assumption of non-negative fees in the main model and allow the exchanges to set negative fees (i.e. paying the advertisers to join), then in equilibrium the fees become negative down to the point where the exchanges make zero total profit. In practice, these type of equilibria do not usually occur because there is some differentiation between exchanges that prevents the Bertrand-type competition. In this section, we extend our model to consider horizontally differentiated exchanges and allow negative buyer-side fees in order to investigate the fee structure that results from these changes. The goal is to establish the robustness of the main results of the paper under this extension.

To model horizontal differentiation, we use a standard loyal/switcher model (e.g., see Narasimhan, 1988; Iyer et al., 2005). A loyal advertiser to an exchange is an advertiser who never goes to the other exchange; i.e., he either goes to the exchange he is loyal to, or to none of the exchanges (if his expected payoff from joining the exchange he is loyal to is negative). A switcher is an advertiser who can go to either exchange depending on his expected payoff.

For illustrative purposes for four advertisers, we assume that each exchange has one loyal advertiser and there are two switchers who can choose between the exchanges. We assume that the buyer-side fees can potentially be negative, i.e. the exchanges can pay the advertisers to join them (but of course they only do it if the final expected utility is positive). We analyze the waterfalling and the header bidding games in Sections B.1.1 and B.1.2, respectively. The proofs of the propositions are available in Web Appendix C.5.

B.1.1 Waterfalling

Similarly to other papers that use the loyal/switchers model, the game does not have a pure-strategy equilibrium. This is because on one hand the exchanges want to lower the fees to attract the switchers but on the other hand they want to increase the fees to extract more surplus from the loyal advertisers. We present a mixed-strategy equilibrium in Proposition 8.

Proposition 8. *Under waterfalling, there is a mixed equilibrium where Exchange 1 (the first one in the waterfall) with probability* $p = \frac{27(6f+5)}{8(708f+125)}$ *sets the buyer-side fee to be* $f_1 = \frac{225}{8192}$*, while with probability* $1-p$ *it chooses a fee drawn from the distribution with CDF*

$$G_1(x) = \frac{72(708f+125)(917f+8192x-75)}{(5502f+865)(8496f+49152x+185)}$$

and domain $\left[\frac{75-917f}{8192}, \frac{1315}{49152}\right)$*, where* f *is the seller-side fee. Exchange 2 with probability* $q = \frac{35}{16506f+2630}$ *sets the buyer-side fee to be* $f_2 = \frac{125}{4096}$ *while with probability* $1-q$ *it chooses a fee drawn from the distribution with CDF*

$$G_2(x) = \frac{(8253f+1315)(5502f+49152x-635)}{(5502f+865)(8253f+49152x-185)}$$

and domain $\left[\frac{635-5502f}{49152}, \frac{125}{4096}\right)$*. In equilibrium, exchanges are indifferent between first-price and second-price auctions.*

Note that in contrast to Example 1, in the equilibrium described by Proposition 8 both exchanges have a positive number of advertisers for any realization of the fees.

B.1.2 Header Bidding

In the following proposition we describe the mixed equilibrium for the buyer-side fees under header bidding. In contrast to the waterfalling case, here we have a symmetric mixed equilibrium.

Proposition 9. *Under header bidding, there is a mixed equilibrium where both exchanges choose a fee drawn from the distribution with CDF* $G(x) = \frac{98f+960x-13}{98f+640x}$ *and domain* $\left[\frac{13-98f}{960}, \frac{13}{320}\right]$. *In equilibrium, both exchanges use first-price auctions.*

Propositions 8 and 9 show the robustness of our results when exchanges are horizontally differentiated and the fees are allowed to be negative. We see that in equilibrium while the exchanges are indifferent between first-price and second-price auctions under waterfalling, they uniquely use first-price auctions under header bidding. Proposition 9 also shows that, when the exchanges are horizontally differentiated, their buyer-side revenue does not decline all the way to zero under header bidding.

B.2 Multiple Segments of Advertisers

In Section B.1, we consider a loyal/switcher model of horizontally differentiated exchanges and show that under header bidding both exchanges will end up with a positive number of advertisers and will use first-price auctions in equilibrium (in contrast to Proposition 7 where there is an equilibrium where all advertisers join the same exchange and the auction format does not matter).

In this section, we consider a more general model of horizontally differentiated exchanges and show that even under very mild assumptions of horizontal differentiation, the monopolistic equilibrium where all advertisers join the same exchange is eliminated. As a result, exchanges uniquely use first-price auctions in equilibrium.

In practice, advertisers can have an intrinsic preference for one exchange,[37] but they can also go to another exchange if the benefit is sufficiently high. To capture this intuition, we assume that there are three segments of advertisers: Segment 1, Segment 2, and Segment 0. Segment-1 advertisers have a slight preference $\delta > 0$ for joining Exchange 1 over Exchange 2. For example, a Segment-1 advertiser will choose Exchange 1 as long as his payoff by moving to Exchange 2 will not increase by more than δ. Similarly, Segment-2 advertisers have a slight preference $\delta > 0$ for joining Exchange 2 over Exchange 1. Finally, Segment-0 advertisers do not have any intrinsic preference for an exchange and if everything else is equal they are indifferent between the two exchanges.

In Section B.2.1, we consider this setting under the timeline of the main model (Section 2.1) and the timeline of Section A.2.1. In Section C.7 (Web Appendix), we consider it under the timeline of Section A.2.2.

B.2.1 Advertisers choose exchanges before the auction format decision

In this section, we consider the two timelines of Sections 2.1 and A.2.1. We show that under both timelines, as long as Segments 1 and 2 are non-empty, the following proposition holds.

Proposition 10. *For any $\delta > 0$, both exchanges have a positive number of advertisers and use first-price auctions in equilibrium.*

Proposition 10 shows that as long as each exchange is preferred by at least one advertiser, even if the preference is infinitesimal, we get a unique equilibrium in which both exchanges use first-price auctions. In other words, even a mild assumption of horizontal differentiation eliminates the equilibrium where all advertisers join the same exchange and the exchange is indifferent between the auction formats.

The proof of Proposition 10 is available in Web Appendix C.6.

[37] For example, an exchange may offer specialized analytics tools that are more valuable to some advertisers than to others. Also, some advertisers might have already made infrastructure investments with an exchange and the cost of switching to a different exchange may be non-negligible.

C Web Appendix

C.1 Equilibrium Refinement for Proposition 2

Proposition 2 shows that the exchanges can obtain positive buyer-side revenue in at least some equilibria of the game. In the following, we show that under some equilibrium refinement assumptions, the exchanges can obtain positive buyer-side revenue in *all* equilibria of the game.

Equilibrium Refinement For given fees f_1 and f_2, there can be several different subgame equilibria, i.e. pair of values n_1, n_2 such that no advertiser has incentive to move to a different exchange. This results in multiple possible equilibria for the whole game, based on what rule the advertisers use to choose an exchange in each scenario. To avoid some of the more "unintuitive" equilibria, we can use the following equilibrium refinement.

Definition 1. The set of values (f_1, f_2, n_1, n_2) is a refined equilibrium iff both of the following conditions are true.

1. No advertiser is better off by moving to another exchange. Formally,[38]

$$n_1 \geq 1 \Rightarrow u_1(n_1, n_2) - f_1 \geq u_2(n_1 - 1, n_2 + 1) - f_2 \qquad \text{and}$$
$$n_2 \geq 1 \Rightarrow u_2(n_1, n_2) - f_2 \geq u_1(n_1 + 1, n_2 - 1) - f_1.$$

2. No exchange has a *profitable deviation*.

Definition 1 requires another definition of what a *profitable deviation* for an exchange is. We define it as follows.

Definition 2. Given the set of values (f_1, f_2, n_1, n_2), a move by Exchange 1 from f_1 to $f_1' = f_1 + \epsilon$ for an $\epsilon > 0$ is considered a profitable deviation iff both of the following conditions are true.

[38]The expressions for u_1 and u_2 are given in (1) and (2).

1. In (f_1', f_2, n_1, n_2), Exchange 1 is *strictly* better off compared to (f_1, f_2, n_1, n_2).[39]

2. In (f_1', f_2, n_1, n_2), no advertiser is better off by moving to another exchange.

A move by Exchange 1 from f_1 to $f_1' = f_1 - \epsilon$ for a sufficiently small $\epsilon > 0$ is considered a profitable deviation iff the following condition is true.

1. In (f_1', f_2, n_1, n_2), at least one advertiser is better off by moving from Exchange 2 to Exchange 1.[40]

We define the profitable deviation for Exchange 2 similarly.

Given Definition 1, we can now prove Proposition 11.

Proposition 11. *Under waterfalling, total exchange revenue through the buyer-side fees is always positive in any refined equilibrium of the game, i.e.* $n_1 f_1 + n_2 f_2 > 0$.

Proof. We need to eliminate three possible equilibria.

1. $(0, f_2, n_1, 0)$ with $n_1 > 0$:

   This is not an equilibrium because Exchange 1 has a profitable deviation. More specifically, there is a sufficiently small $\epsilon > 0$ such that in $(\epsilon, f_2, n_1, 0)$ no advertiser is better off by moving to Exchange 2. The value of ϵ can be up to $u_1(n_1, 0) - u_2(n_1 - 1, 1) + f_2 > 0$.

2. $(f_1, 0, 0, n_2)$ with $n_2 > 0$:

   This is not an equilibrium because now Exchange 2 has a profitable deviation, similar to the first case. More specifically, Exchange 2 can charge a fee up to $u_2(0, n_2) - u_1(1, n_2 - 1) + f_1 > 0$, so that no advertiser wants to move to Exchange 1.

3. $(0, 0, n_1, n_2)$ with $n_1, n_2 > 0$:

[39] Equivalently, $n_1 > 0$.

[40] Intuitively, Exchange 1 decreases its fee slightly but it gets at least one extra advertiser. As a result, it increases its overall revenue.

This is not an equilibrium, because advertisers can be better off by moving to a different exchange. It is easy to verify that the inequalities $u_1(n_1, n_2) \geq u_2(n_1 - 1, n_2 + 1)$ and $u_2(n_1, n_2) \geq u_1(n_1 + 1, n_2 - 1)$ are not simultaneously true for any $n_1, n_2 > 0$ with $n_1 + n_2 \leq n$.

As a result, in any refined equilibrium of the game, we have $n_1 f_1 + n_2 f_2 > 0$. □

In Example 1, we can see a refined equilibrium where the total buyer-side revenue is positive.

C.2 Proofs of Section A.2

Proof of Proposition 6. Under this timeline, the publisher will set the reserve prices based on the auction formats chosen by the exchanges. If both exchanges choose a first-price auction, then similarly to the main model, the publisher will set the reserve prices to $r_1 = r_2 = \frac{1}{2}$ for optimal revenue. There are two remaining cases to consider.

Case I. In this case one exchange is using a first-price auction while the other is using a second-price auction. Without loss of generality we assume that Exchange 1 is using a first-price auction, while Exchange 2 is using a second-price auction. We consider six subcases.

1. $n_1 = 2$ and $n_2 = 2$. To find out the optimal reserve prices, first we need to find out the bidding strategy of the advertisers in Exchange 1.

   To find out the bidding strategy, consider the following intuition. If the reserve price r_2 is sufficiently larger than r_1, then Exchange 1's advertisers with relatively low valuations (but above r_1) know that they cannot compete with the winner of Exchange 2, therefore their best course of action is to ignore Exchange 2 and bid as if there is only one exchange (hoping that noone in Exchange 2 will be above r_2). However, Exchange 1's advertisers with high valuations will want to compete with Exchange 2 and their bidding function will consider both exchanges. Therefore, in general if $r_2 \geq r_1$, the bidding function of Exchange 1's advertisers is of the form

$$b_{2,2}^{r_2 \geq r_1}(v) = \begin{cases} 0 & \text{, if } v \leq r_1, \\ v - \frac{v}{n_1} + \frac{r^{n_1}}{n_1 \cdot v^{n_1 - 1}} & \text{, if } r_1 < v \leq t, \\ b(v) & \text{, if } v > t, \end{cases}$$

where $b(v)$ is the solution to the differential equation $\frac{\partial u}{\partial x}\Big|_{x=v} = 0$ with boundary condition $b(t) = r_2$.[41] The threshold t is determined as the point where the utility from both bidding functions is the same (or r_1; whichever is larger).

If $r_1 > r_2$, then Exchange 1's advertisers will always consider Exchange 2, and their bidding function will be of the form

$$b_{2,2}^{r_1 > r_2}(v) = \begin{cases} 0 & \text{, if } v \leq r_1, \\ b(v) & \text{, if } v > r_1, \end{cases}$$

where $b(v)$ is the solution to the differential equation $\frac{\partial u}{\partial x}\Big|_{x=v} = 0$ with boundary condition $b(r_1) = r_1$.

Now that we know the form of the bidding strategies, for any given r_1 and r_2 we can find out the bidding functions and the publisher's revenue. Therefore, we can numerically find the optimal values of r_1 and r_2. These are $r_1^* \approx 0.472$ and $r_2^* \approx 0.666$. Publisher's revenue for these reserve prices is $\approx (1 - f) \cdot 0.59099$. Exchange 1's revenue is $\approx f \cdot 0.32$, while Exchange 2's revenue is $\approx f \cdot 0.27$. Exchange 1's advertisers bidding function is plotted in Figure W.1.

2. $n_1 = 3$ and $n_2 = 1$. In this subcase, the bidding function of advertisers in Exchange 1 has a similar form to the one in the previous subcase. Doing a similar analysis, we can find out that the optimal reserve prices are $r_1^* \approx 0.493$ and $r_2^* \approx 0.677$. Publisher's

[41] u is advertiser's utility function as given in (3).

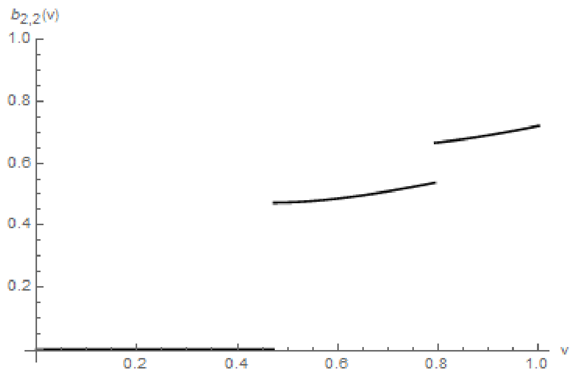


Figure W.1: Bidding function of advertisers in Exchange 1 for $n_1 = 2$ and $n_2 = 2$, when Exchange 1 is using a first-price auction, Exchange 2 is using a second-price auction, $r_1 \approx 0.472$, and $r_2 \approx 0.666$.

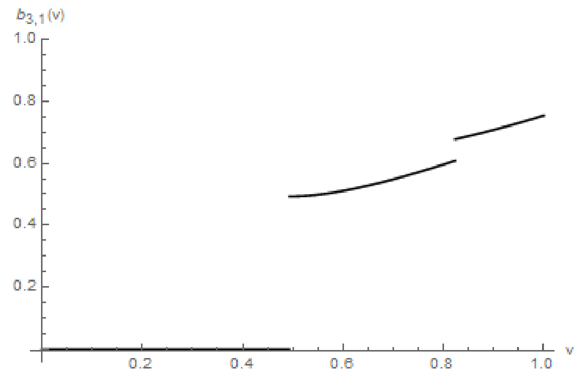


Figure W.2: Bidding function of advertisers in Exchange 1 for $n_1 = 3$ and $n_2 = 1$, when Exchange 1 is using a first-price auction, Exchange 2 is using a second-price auction, $r_1 \approx 0.493$, and $r_2 \approx 0.677$.

revenue for these reserve prices is $\approx (1-f)\cdot 0.6008$. Exchange 1's revenue is $\approx f \cdot 0.48$, while Exchange 2's revenue is $\approx f \cdot 0.12$. Exchange 1's advertisers bidding function is plotted in Figure W.2.

3. $n_1 = 1$ and $n_2 = 3$. In this subcase there is only one advertiser in Exchange 1. If his valuation v is sufficiently low (but above r_1), then he will ignore Exchange 2 and he will just bid $r_1 + \epsilon$ for an $\epsilon \in (0, v - r_1)$. If his valuation is high, then he will consider Exchange 2 and his bid will be $z(v) = \frac{1}{16}\left(6v + 9 - \sqrt{36v^2 - 84v + 81}\right)$ (i.e., the point y that maximizes the function $u(b^{-1}(y))$). Therefore, his bidding function will be of the form

$$b(v) = \begin{cases} 0 & \text{, if } v \leq r_1, \\ r + \epsilon & \text{, if } r_1 < v < t, \\ \frac{1}{16}\left(6v + 9 - \sqrt{36v^2 - 84v + 81}\right) & \text{, if } v \geq t, \end{cases}$$

where the threshold t that determines what bidding strategy to follow is the point where $z(t) = \max\{r_1, r_2\}$.

Given this bidding strategy, we can find out that the optimal reserve prices are $r_1^* \approx 0.734$ and $r_2^* = 0.5$. Publisher's revenue for these reserve prices is $\approx (1-f)\cdot 0.589522$.

Exchange 1's revenue is $\approx f \cdot 0.16$, while Exchange 2's revenue is $\approx f \cdot 0.43$.

4. $n_1 = 2$ and $n_2 = 1$. This is similar to subcase 1. The optimal reserve prices are $r_1^* \approx 0.487$ and $r_2^* \approx 0.639$. Publisher's revenue for these reserve prices is $\approx (1-f) \cdot 0.518705$. Exchange 1's revenue is $\approx f \cdot 0.37$, while Exchange 2's revenue is $\approx f \cdot 0.15$.

5. $n_1 = 1$ and $n_2 = 2$. This is similar to subcase 3. The optimal reserve prices are $r_1^* \approx 0.688$ and $r_2^* = 0.5$. Publisher's revenue for these reserve prices is $\approx (1 - f) \cdot 0.504481$. Exchange 1's revenue is $\approx f \cdot 0.19$, while Exchange 2's revenue is $\approx f \cdot 0.31$.

6. $n_1 = 1$ and $n_2 = 1$. This is similar to subcase 1. The optimal reserve prices are $r_1^* \approx 0.472$ and $r_2^* = 0.593$. Publisher's revenue for these reserve prices is $\approx (1-f) \cdot 0.40569$. Exchange 1's revenue is $\approx f \cdot 0.22$, while Exchange 2's revenue is $\approx f \cdot 0.19$.

Case II. In this case both exchanges are using a second-price auction. Since all advertisers are bidding truthfully, to find out the optimal reserve prices we need to solve a simple optimization problem. We consider four subcases.

1. $n_1 = 3$ and $n_2 = 1$. The optimal reserve prices are $r_1^* = 0.5$ and $r_2^* \approx 0.734$. Publisher's revenue for these reserve prices is $\approx (1-f) \cdot 0.589522$. Exchange 1's revenue is $\approx f \cdot 0.43$, while Exchange 2's revenue is $\approx f \cdot 0.16$.

2. $n_1 = 2$ and $n_2 = 2$. In this subcase, there are two asymmetric equilibria that give the same revenue to the publisher. The two equilibria are given for the reserve prices $(r_1^*, r_2^*) \approx (0.5, 0.688)$ and $(r_1^*, r_2^*) \approx (0.688, 0.5)$. Publisher's revenue for these reserve prices is $\approx (1 - f) \cdot 0.57443$. The revenue of the exchange with the smallest reserve price is $\approx f \cdot 0.23$, while the revenue of the exchange with the largest reserve price is $\approx f \cdot 0.34$.

   Since both sets of reserve prices are optimal for the publisher, we assume that the publisher picks one at random (with probability 0.5). Therefore, the expected revenue of each exchange in this subcase is $\approx f \cdot 0.285$.

3. $n_1 = 2$ and $n_2 = 1$. The optimal reserve prices are $r_1^* = 0.5$ and $r_2^* \approx 0.688$. Publisher's revenue for these reserve prices is $\approx (1-f)\cdot 0.504481$. Exchange 1's revenue is $\approx f \cdot 0.31$, while Exchange 2's revenue is $\approx f \cdot 0.19$.

4. $n_1 = 1$ and $n_2 = 1$. Here there are two asymmetric equilibria as well. The optimal reserve prices are 0.5 and 0.625. Publisher's revenue for these reserve prices is $\approx (1-f) \cdot 0.390625$. The revenue of the exchange with the smallest reserve price is $\approx f \cdot 0.16$, while while the revenue of the exchange with the largest reserve price is $\approx f \cdot 0.23$. Therefore, the expected revenue of each exchange is $\approx f \cdot 0.195$.

Summarizing all of the above, the payoff matrix of the game between the exchanges is as follows.

- For $n_1 = 2$ and $n_2 = 2$, it is

$f\cdot$	$\{0.285, 0.285\}$	$\{0.27, 0.32\}$
	$\{0.32, 0.27\}$	$\{0.30625, 0.30625\}$

- For $n_1 = 1$ and $n_2 = 3$, it is

$f\cdot$	$\{0.16, 0.43\}$	$\{0.12, 0.48\}$
	$\{0.16, 0.43\}$	$\{0.153125, 0.459375\}$

- For $n_1 = 1$ and $n_2 = 2$, it is

$f\cdot$	$\{0.19, 0.31\}$	$\{0.15, 0.37\}$
	$\{0.19, 0.31\}$	$\{0.177083, 0.354167\}$

- For $n_1 = 1$ and $n_2 = 1$, it is

$f\cdot$	$\{0.195, 0.195\}$	$\{0.19, 0.22\}$
	$\{0.22, 0.19\}$	$\{0.208333, 0.208333\}$

As we can see, in all cases there is a unique equilibrium in which both exchanges use first-price auctions. □

Proof of Proposition 7. If both exchanges use a first-price auction, then no matter how advertisers split between the two exchanges, the publisher will set both reserve prices to be $\frac{1}{2}$ and publisher's revenue will be optimal (i.e. equal to $(1-f)\cdot 0.6125$). This also means that advertisers do not care which exchange they join and we can assume that they pick an exchange at random. Therefore, an exchange's expected revenue will be $f \cdot 0.30625$.

If both exchanges use a second-price auction, then we will show that all advertisers will join a single exchange. We consider three cases.

- Suppose that all 4 advertisers join Exchange 1. The publisher will then set a reserve price of $r_1 = \frac{1}{2}$. Each advertiser will have an expected utility of $\frac{31}{160} - \frac{1}{4} \cdot \frac{49}{80} = 0.040625$.

- Suppose that 3 advertisers join Exchange 1 and 1 advertiser joins Exchange 2. The publisher will set the reserve prices to be $r_1 = \frac{1}{2}$ and $r_2 \approx 0.734$. The expected utility of an advertiser in Exchange 1 will be $\approx 0.186179 - \frac{1}{3} \cdot 0.43 = 0.0428457$. The expected utility of the advertiser in Exchange 2 will be $\approx 0.190349 - 0.16 = 0.030349$.

- Suppose that each exchange has 2 advertisers. The publisher will set one reserve price to $\frac{1}{2}$ and another to ≈ 0.688. Without loss of generality, we assume that $r_1 = \frac{1}{2}$ and $r_2 \approx 0.688$. The expected utility of an advertiser in Exchange 1 will be $\approx 0.158953 - \frac{1}{2} \cdot 0.23 = 0.043953$. The expected utility of an advertiser in Exchange 2 will be $\approx 0.205071 - \frac{1}{2} \cdot 0.34 = 0.035071$.

  In this case, when advertisers choose an exchange, they don't know if they will get the high reserve price or the low one. Therefore, if we assume that the publisher assigns the two reserve prices at random, then the expected utility of an advertiser will be $\frac{0.043953+0.035071}{2} = 0.039512$.

As we can see, the only case where there is no advertiser who wants to move to a different exchange is when all advertisers are in the same exchange. When this happens, the expected revenue of the exchange with all the advertisers is $f \cdot 0.6125$. Since the exchanges do not know which exchange all the advertisers will go to when they choose the auction format, we can assume that there is an equal probability for both exchanges. Therefore, the expected utility of both exchanges when they both use a second-price auction will be $\frac{f \cdot 0.6125}{2} = f \cdot 0.30625$ (the same as when both exchanges use a first-price auction).

Finally, we consider the case where one exchange is using a first-price auction and the other a second-price auction. The optimal expected clearing price for the publisher is 0.6125 (and it can be achieved if both exchanges use a first-price auction or if all advertisers join the same exchange). If the two exchanges use a different auction format and both have a positive number of advertisers, then the expected final clearing price will be < 0.6125. Therefore, at least one exchange will have an expected revenue of $< \frac{f \cdot 0.6125}{2} = f \cdot 0.30625$. That exchange will want to change its auction format to match the one of the other exchange, therefore we cannot have an equilibrium of this form. □

C.3 Large number of advertisers

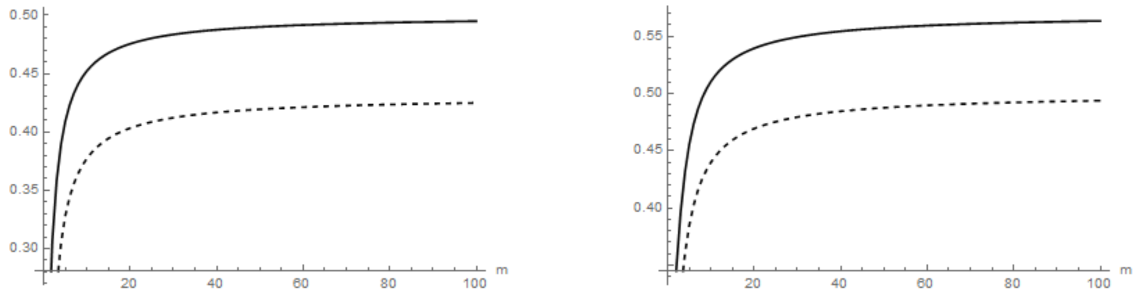
In this section, we consider the case where there are more than 4 advertisers in total.

Almost all of the results in the paper can easily be generalized for any number of advertisers, as we can see from the proofs, except for Proposition 4. To prove Proposition 4 for any number of advertisers in the two exchanges, we need to know the bidding function of advertisers in a first-price auction when the competing exchange is using a second-price auction. Even though we know the differential equation that this bidding function satisfies (see for example Equation 4 in Lemma 3 for the case of $n_1 = n_2 = 2$.), we do not have a closed-form solution of it for general n_1 and n_2.[42]

However, what we can do is to numerically show that the result holds for different values of n_1 and n_2. Since we have the differential equation that the bidding function satisfies, we can solve it numerically for any given n_1, n_2 and then we can use the numerical solution to find the equilibrium of the game.

First, we consider two symmetric exchanges with m advertisers in each one. For $m =$

[42] It is also possible that a general closed-form solution does not exist.



(a) $L_1^{(FP,FP)} > L_1^{(SP,FP)}$ (solid line) (dashed line)

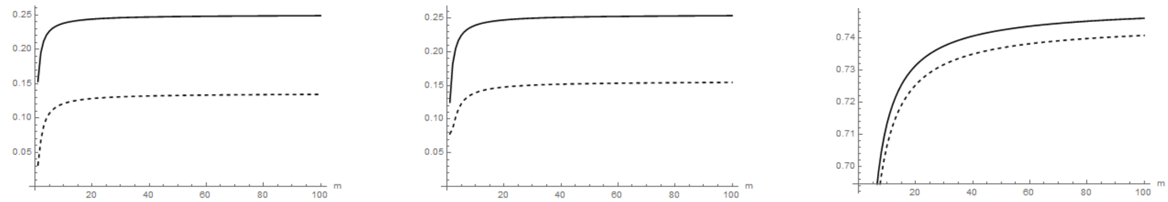
(b) $L_1^{(FP,SP)} > L_1^{(SP,SP)}$ (solid line) (dashed line)

Figure W.3: Exchanges' revenues for different choices of auction formats, for $n_1 = n_2 = m$, and $m = 1, \ldots, 100$.

$1, \ldots, 100$, we will numerically test if in equilibrium both exchanges use first-price auctions (and if there is any other equilibrium). In Figure W.3, we can see the expected revenues of the exchanges (ignoring the fees) for the different values of m. The solid line in Figure W.3a corresponds to the revenue of an exchange when both exchanges are using first-price auctions. Using the notation of the proof of Proposition 4, this is $L_1^{(FP,FP)}$ (which due to symmetry is the same as $L_2^{(FP,FP)}$). The dashed line in Figure W.3a corresponds to $L_1^{(SP,FP)}$ (which due to symmetry is the same as $L_2^{(FP,SP)}$). The solid line in Figure W.3b corresponds to $L_1^{(FP,SP)}$, while the dashed line corresponds to $L_1^{(SP,SP)}$. Since the solid lines are above the dashed lines in both plots for any m, we conclude that (FP, FP) is the unique equilibrium in all cases.

Next, we consider two exchanges with $n_1 = m$ and $n_2 = 3m$ advertisers, for $m = 1, \ldots, 100$. In Figure W.4a, the solid line corresponds to $L_1^{(FP,FP)}$, while the dashed line corresponds to $L_1^{(SP,FP)}$. In Figure W.4b, the solid line corresponds to $L_1^{(FP,SP)}$, while the dashed line corresponds to $L_1^{(SP,SP)}$. In Figure W.4c, the solid line corresponds to $L_2^{(FP,FP)}$, while the dashed line corresponds to $L_2^{(FP,SP)}$. In all plots, the solid lines are above the dashed lines for any m, therefore (FP, FP) is the unique equilibrium in all cases.

Finally, in the extreme case where all advertisers join one exchange, e.g. when $n_1 = 0$, $n_2 > 0$, the auction formats do not matter because of the revenue equivalence principle. Therefore, the exchanges can choose a first or a second-price format and the outcome will



(a) $L_1^{(FP,FP)} > L_1^{(SP,FP)}$ (solid line) (dashed line)

(b) $L_1^{(FP,SP)} > L_1^{(SP,SP)}$ (solid line) (dashed line)

(c) $L_2^{(FP,FP)} > L_2^{(FP,SP)}$ (solid line) (dashed line)

Figure W.4: Exchanges' revenues for different choices of auction formats, for $n_1 = m$, $n_2 = 3m$, and $m = 1, \ldots, 100$.

be the same (e.g. $L_2^{(FP,FP)} = L_2^{(FP,SP)}$).

C.4 Endogenous decision of exchange ordering in waterfalling

In this section, we consider the case where the ordering of the exchanges under waterfalling is an endogenous decision of the publisher. More specifically, we assume that the publisher decides the order of the exchanges after the advertisers join an exchange and the exchanges decide their auction format.

As we saw in Proposition 1, the auction format does not affect the outcome of the game under waterfalling. Therefore, without loss of generality, we can assume that both exchanges are using second-price auctions. We consider three cases.

- One exchange has 3 advertisers and the other exchange has 1 advertiser. The publisher has to decide the order of the two exchanges in the waterfall. We consider two subcases.
  - If $n_1 = 3$ and $n_2 = 1$, then using the reserve prices from Lemma 2, we can find that publisher's revenue will be $\approx (1 - f) \cdot 0.576294$.
  - If $n_1 = 1$ and $n_2 = 3$, then similarly we can calculate that publisher's revenue will be $\approx (1 - f) \cdot 0.586182$.

  Therefore, the publisher will go to the smallest exchange with 1 advertiser first in the waterfall. In this case, using the formulas from the proof of Proposition 2, we can see that the advertiser in the first exchange will have an expected utility of $\approx$

0.0274658, while each advertiser in the second exchange will have an expected utility of ≈ 0.0438639.

- Each exchange has 2 advertisers. In this case, the two exchanges are symmetric, therefore the order does not matter for the publisher. Each advertiser in the first exchange will have an expected utility of ≈ 0.0342641, while each advertiser in the second exchange will have an expected utility of ≈ 0.0418113. As we can see, in this case, the advertisers in the second exchange are better off than the ones in the first exchange.

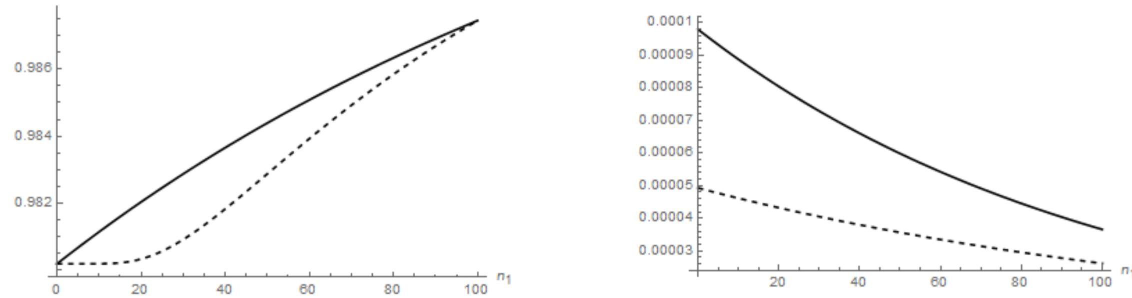
- All 4 advertisers join the same exchange. In this case again the order does not matter for the publisher, since only one exchange has advertisers. Each advertiser will have an expected utility of ≈ 0.040625.

From the above, we can conclude that there is only one equilibrium in the game. In equilibrium all advertisers will join the same exchange, since in any other case there is at least one advertiser who can improve his utility by changing exchange. Moreover, the exchange with all the advertisers can charge a fee of up to $\approx 0.040625 - 0.0274658 = 0.0131592$ and no advertiser will want to move to the other exchange. This is the same equilibrium as the one in Example 1, therefore endogenizing the decision of the exchange ordering does not change the equilibrium result.

Larger number of advertisers. Finally, we can check if this result is true for a larger number of advertisers. More specifically, we can numerically verify the following two observations.

- For $n_1 \leq n_2$, the publisher will place Exchange 1 first in the waterfall to maximize revenue.

  Let $\text{WF}_{1,2}$ be publisher's revenue when he places Exchange 1 first in the waterfall, and $\text{WF}_{2,1}$ his revenue when he places Exchange 2 first. In Figure W.5a (where



(a) $\mathrm{WF}_{1,2} \geq \mathrm{WF}_{2,1}$ (solid line) (dashed line)

(b) $u_2(n_1-1, n_2+1) > u_1(n_1, n_2)$ (solid line) (dashed line)

Figure W.5: Waterfalling with $n_2 = 100$ and $n_1 \leq n_2$

$n_1 \leq n_2 = 100$), we can see that $\mathrm{WF}_{1,2} \geq \mathrm{WF}_{2,1}$, with equality when all advertisers are in one exchange or the two exchanges have the same number of advertisers.

- For $n_1 \leq n_2$, we get $u_2(n_1-1, n_2+1) > u_1(n_1, n_2)$, i.e. an advertiser in the smallest exchange has incentive to move to the largest exchange.

  In Figure W.5b we can see an illustration of this for $n_2 = 100$ (it is actually true even for some values $n_1 > n_2$, because there is an extra benefit for being second in the waterfall due to the lower reserve price).

From these two observations, we can conclude that (ignoring buyer-side fees) there is a unique subgame equlibrium under waterfalling, where all advertisers choose the same exchange (similarly to Example 1).

C.5 Proofs of Section B.1

Proof of Proposition 8. Let $u_i(n_1, n_2)$ be the expected utility of an advertiser in Exchange i (ignoring the fee) and $R_i(n_1, n_2)$ be the seller-side revenue of Exchange i, when Exchange 1 has n_1 advertisers and Exchange 2 has n_2 advertisers.

For two fixed buyer-side fees f_1 and f_2, it is possible that there are multiple subgame equilibria, e.g. one where both switchers go to Exchange 1 and one where both switchers go to Exchange 2. The selection rule we use in that case is choosing the subgame equilibrium which is best for the advertisers (e.g. we can assume that the switchers coordinate to go to the exchange that maximizes their payoff). Let $t = u_2(1,3) - u_1(3,1) = \frac{185}{49152}$. Note that if the two loyals are in their own exchange and $f_1 - f_2 < t$, the two switchers prefer to be in Exchange 1. Otherwise if $f_1 - f_2 > t$, the two switchers prefer to be in Exchange 2. Note also that for the bounds of the two domains of the CDFs G_1 and G_2, it holds that $\frac{635-5502f}{49152} - \frac{75-917f}{8192} = \frac{125}{4096} - \frac{1315}{49152} = t$.

If Exchange 1 sets a fee $x \in \left[\frac{75-917f}{8192}, \frac{1315}{49152}\right)$, then its expected profit is

$$(1-q)G_2(x+t)(x+R_1(1,3)) + ((1-q)(1-G_2(x+t))+q)(3x+R_1(3,1)) = \frac{735f}{4096} + \frac{225}{8192},$$

which is independent of x. If Exchange 1 sets the fee $f_1 = u_1(1,3) = \frac{225}{8192}$, then its expected profit is also

$$u_1(1,3) + R_1(1,3) = \frac{735f}{4096} + \frac{225}{8192}.$$

Therefore, Exchange 1 is indifferent among the fees in $\left[\frac{75-917f}{8192}, \frac{1315}{49152}\right) \cup \left\{\frac{225}{8192}\right\}$.

Any fee $x < \frac{75-917f}{8192}$ is strictly worse than $\frac{75-917f}{8192}$, because in both cases Exchange 1 gets 3 advertisers but in the first case they pay a lower fee. Any fee in $\left[\frac{1315}{49152}, \frac{225}{8192}\right)$ is dominated by the fee $\frac{225}{8192}$, because in both cases Exchange 1 gets only the loyal advertiser, but in the first case he pays a lower fee. Finally, any fee in $\left[\frac{225}{8192}, +\infty\right)$ is strictly worse, because in this case Exchange 1 gets no advertisers. Therefore, Exchange 1 has no incentive to deviate from its strategy.

If Exchange 2 sets a fee $x \in \left[\frac{635-5502f}{49152}, \frac{125}{4096}\right]$, then its expected profit is

$$(1-p)G_1(x-t)(x+R_2(3,1)) + ((1-p)(1-G_1(x-t))+p)(3x+R_2(1,3)) = \frac{1162f+635}{16384},$$

which is independent of x, i.e. Exchange 2 is indifferent among the fees in $\left[\frac{635-5502f}{49152}, \frac{125}{4096}\right]$.

Any fee $x < \frac{635-5502f}{49152}$ is strictly worse than $\frac{635-5502f}{49152}$, because in both cases Exchange 2 gets 3 advertisers but in the first case they pay a lower fee. It remains to show that Exchange 2 has no incentive to set a fee $x > \frac{125}{4096}$. The largest fee it can set while still having a chance to get some advertisers is $\frac{225}{8192} + t = \frac{1535}{49152}$. For this fee, its expected payoff is

$$(1-p) \cdot 0 + p\left(3 \cdot \frac{1535}{49152} + R_2(1,3)\right) = \frac{27(6f+5)(6664f+1535)}{131072(708f+125)}.$$

For $f \in [0,1]$, this is strictly smaller than Exchange 2's payoff, $\frac{1162f+635}{16384}$. Finally, any fee in $\left(\frac{125}{4096}, \frac{1535}{49152}\right)$ is dominated by the fee $\frac{1535}{49152}$, which is worse than any fee in the support of Exchange 2's strategy. Therefore, Exchange 2 does not want to deviate either. □

Proof of Proposition 9. For the given fee structure, we will see that both exchanges get a positive number of advertisers for any realization of the fees. Therefore, from Proposition 4 we know that in the subgame after the buyer-side fees have been fixed, both exchanges will choose to use first-price auctions. It remains to show that the given fee structure is an equilibrium for the full game.

Let u be the expected utility of an advertiser (ignoring fees) for participating in the auction and r be the seller-side revenue of an exchange per advertiser, when both exchanges use first-price auctions. These values are $u = \frac{13}{320}$ and $r = \frac{49f}{320}$.

If Exchange 1 sets a fee $x \in \left[\frac{13-98f}{960}, \frac{13}{320}\right]$, then its expected profit is

$$G(x)(x+r) + (1-G(x))(3x+3r) = \frac{49f+13}{320},$$

which is independent of x, therefore Exchange 1 is indifferent among the fees in $\left[\frac{13-98f}{960}, \frac{13}{320}\right]$.

Any fee $x < \frac{13-98f}{960}$ is strictly worse than $\frac{13-98f}{960}$, because in both cases Exchange 1 gets 3 advertisers but in the first case they pay a lower fee. Also, any fee $x > u = \frac{13}{320}$ is worse because then the exchange gets no advertisers. Therefore, Exchange 1 has no incentive to deviate from its strategy. Due to symmetry, the same is true for Exchange 2. □

C.6 Proofs of Section B.2

Proof of Proposition 10. Suppose that all advertisers are in the same exchange. If one advertiser moves to the other exchange, then from Propositions 4 and 6 (depending on the timeline) we know that both exchanges will be using first-price auctions. When this happens, the utility for the advertiser who moved will be the same as in a unified auction, if we ignore the segments. Given this indifference, for any $\delta > 0$, a Segment-1 or Segment-2 advertiser will always want to be in his exchange. Since Segments 1 and 2 are non-empty, we cannot have an equilibrium where all advertisers are joining the same exchange. The result follows from Propositions 4 and 6, depending on the timeline we are in. □

C.7 Multiple Segments of Advertisers (Alternative Timeline)

This is a continuation of Section B.2. In this section we consider the same model, under the timeline of Section A.2.2, where advertisers choose exchanges after the auction format decision. We show that as long as Segments 1 and 2 are non-empty, the following proposition holds.

Proposition 12. *For any $\delta > \overline{\delta}$, both exchanges have a positive number of advertisers and use first-price auctions in equilibrium, where*

$$\overline{\delta} = \frac{2^{n+1} - n - 2}{n(n+1)2^{n+1}} - \frac{(n(1-r_1) + 2r_1 - 1)(1-r_1)^2 r_1^{n-2}}{2}$$

and r_1 is the root in $[0,1]$ of the polynomial $r_1^{n-2}(1-r_1)^2(n-1) + \frac{1-2^n r_1^n}{2^{n-1}n}$.

Proposition 12 shows that as long as each exchange is preferred by at least one advertiser, if the preference is sufficiently high,[43] we get a unique equilibrium in which both exchanges use first-price auctions. In other words, with a mild assumption of horizontal differentiation, the second equilibrium of Proposition 7, where all advertisers join the same exchange, is eliminated.

[43]Note that for $n = 4$, we get $\overline{\delta} \approx 0.01$.

Proof of Proposition 12. We will do the proof in two steps. First, we will show that the situation where both exchanges are using second-price auctions and all advertisers join the same exchange is not an equilibrium any more (as it was in Proposition 7). Second, we will show that no new equilibrium arises, other than the one where both exchanges use first-price auctions.

Suppose that both exchanges are using second-price auctions and w.l.o.g. all advertisers join Exchange 2. The expected utility of an advertiser in Segment 1 will be $\frac{2^{n+1}-n-2}{n(n+1)2^{n+1}}$. Now if this advertiser moves to Exchange 1, his expected utility will be $\delta + \frac{(n(1-r_1)+2r_1-1)(1-r_1)^2 r_1^{n-2}}{2}$, where r_1 is the reserve price that the publisher will set for Exchange 1 and is given as the root of the polynomial $r_1^{n-2}(1-r_1)^2(n-1) + \frac{1-2^n r_1^n}{2^{n-1}n}$ (which is derived from the first-order condition of the publisher's revenue). Since $\delta > \overline{\delta} = \frac{2^{n+1}-n-2}{n(n+1)2^{n+1}} - \frac{(n(1-r_1)+2r_1-1)(1-r_1)^2 r_1^{n-2}}{2}$, the advertiser wants to move to Exchange 1 and therefore, it is not an equilibrium for all advertisers to be in the same exchange.

Next, we want to show that for $\delta > \overline{\delta}$, where $\overline{\delta}$ is determined as above, the only equilibrium is when both exchanges are using first-price auctions. To show this we can do an analysis similar to the one in the proof of Proposition 7 by considering all the different subgames.

We start by assuming that Segments 1 and 2 have one advertiser each and Segment 0 has two advertisers. In the following table we can see the expected utility of an advertiser in Exchange 1 and in Exchange 2 for different distributions of advertisers between the exchanges and different combinations of auction formats (without including the intrinsic-preference parameter δ). For each column, based on the fact that each exchange is preferred by an advertiser, the highlighted cells determine the subgame equilibria.

Auction Formats

Number of Advertisers	(SP, SP)	(SP, FP)	(FP, SP)	(FP, FP)
$(4, 0)$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$
$(3, 1)$	$\{0.04285, 0.03035\}$	$\{0.04285, 0.03035\}$	$\{0.04301, 0.03081\}$	$\{0.04063, 0.04063\}$
$(2, 2)$	$\{0.03951, 0.03951\}$	$\{0.02893, 0.04911\}$	$\{0.04911, 0.02893\}$	$\{0.04063, 0.04063\}$
$(1, 3)$	$\{0.03035, 0.04285\}$	$\{0.03081, 0.04301\}$	$\{0.03035, 0.04285\}$	$\{0.04063, 0.04063\}$
$(0, 4)$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$

Based on the subgame equilibria highlighted in the table above, we can find the expected revenues of the two exchanges for the different choices of auction formats, as shown in the table below.

Exchange 2

Exchange 1	SP	FP
SP	$f \cdot \{0.295, 0.295\}$	$f \cdot \{0.12, 0.48\}$
FP	$f \cdot \{0.48, 0.12\}$	$f \cdot \{0.30625, 0.30625\}$

As we can see, there is a unique equilibrium, where both exchanges use a first-price auction.

For the remaining cases, we do a similar analysis, as follows.

- Segment 1 has two advertisers, Segment 2 has one advertiser, Segment 0 has one advertiser, and $\delta < 0.01408$.

Auction Formats

Number of Advertisers	(SP, SP)	(SP, FP)	(FP, SP)	(FP, FP)
$(4, 0)$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$
$(3, 1)$	$\{0.04285, 0.03035\}$	$\{0.04285, 0.03035\}$	$\{0.04301, 0.03081\}$	$\{0.04063, 0.04063\}$
$(2, 2)$	$\{0.03951, 0.03951\}$	$\{0.02893, 0.04911\}$	$\{0.04911, 0.02893\}$	$\{0.04063, 0.04063\}$
$(1, 3)$	$\{0.03035, 0.04285\}$	$\{0.03081, 0.04301\}$	$\{0.03035, 0.04285\}$	$\{0.04063, 0.04063\}$
$(0, 4)$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$

Exchange 2

Exchange 1	SP	FP
SP	$f \cdot \{0.43, 0.16\}$	$f \cdot \{0.12, 0.48\}$
FP	$f \cdot \{0.48, 0.12\}$	$f \cdot \{0.3828125, 0.2296875\}$

- Segment 1 has two advertisers, Segment 2 has one advertiser, Segment 0 has one advertiser, and $\delta > 0.01408$.

Auction Formats

Number of Advertisers	(SP, SP)	(SP, FP)	(FP, SP)	(FP, FP)
$(4, 0)$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$
$(3, 1)$	$\{0.04285, 0.03035\}$	$\{0.04285, 0.03035\}$	$\{0.04301, 0.03081\}$	$\{0.04063, 0.04063\}$
$(2, 2)$	$\{0.03951, 0.03951\}$	$\{0.02893, 0.04911\}$	$\{0.04911, 0.02893\}$	$\{0.04063, 0.04063\}$
$(1, 3)$	$\{0.03035, 0.04285\}$	$\{0.03081, 0.04301\}$	$\{0.03035, 0.04285\}$	$\{0.04063, 0.04063\}$
$(0, 4)$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$

Exchange 2

Exchange 1	SP	FP
SP	$f \cdot \{0.43, 0.16\}$	$f \cdot \{0.27, 0.32\}$
FP	$f \cdot \{0.48, 0.12\}$	$f \cdot \{0.3828125, 0.2296875\}$

- Segment 1 has two advertisers, Segment 2 has two advertisers, Segment 0 has no advertiser, and $\delta < 0.01408$.

Auction Formats

Number of Advertisers	(SP, SP)	(SP, FP)	(FP, SP)	(FP, FP)
$(4, 0)$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$
$(3, 1)$	$\{0.04285, 0.03035\}$	$\{0.04285, 0.03035\}$	$\{0.04301, 0.03081\}$	$\{0.04063, 0.04063\}$
$(2, 2)$	$\{0.03951, 0.03951\}$	$\{0.02893, 0.04911\}$	$\{0.04911, 0.02893\}$	$\{0.04063, 0.04063\}$
$(1, 3)$	$\{0.03035, 0.04285\}$	$\{0.03081, 0.04301\}$	$\{0.03035, 0.04285\}$	$\{0.04063, 0.04063\}$
$(0, 4)$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$

Exchange 2

Exchange 1	SP	FP
SP	$f \cdot \{0.285, 0.285\}$	$f \cdot \{0.12, 0.48\}$
FP	$f \cdot \{0.48, 0.12\}$	$f \cdot \{0.30625, 0.30625\}$

- Segment 1 has two advertisers, Segment 2 has two advertisers, Segment 0 has no advertiser, and $\delta > 0.01408$.

Auction Formats

Number of Advertisers	(SP, SP)	(SP, FP)	(FP, SP)	(FP, FP)
$(4, 0)$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$
$(3, 1)$	$\{0.04285, 0.03035\}$	$\{0.04285, 0.03035\}$	$\{0.04301, 0.03081\}$	$\{0.04063, 0.04063\}$
$(2, 2)$	$\{0.03951, 0.03951\}$	$\{0.02893, 0.04911\}$	$\{0.04911, 0.02893\}$	$\{0.04063, 0.04063\}$
$(1, 3)$	$\{0.03035, 0.04285\}$	$\{0.03081, 0.04301\}$	$\{0.03035, 0.04285\}$	$\{0.04063, 0.04063\}$
$(0, 4)$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$

Exchange 2

Exchange 1	SP	FP
SP	$f \cdot \{0.285, 0.285\}$	$f \cdot \{0.27, 0.32\}$
FP	$f \cdot \{0.32, 0.27\}$	$f \cdot \{0.30625, 0.30625\}$

- Segment 1 has three advertisers, Segment 2 has one advertiser, Segment 0 has no advertiser, and $\delta < 0.01408$.

Auction Formats

Number of Advertisers	(SP, SP)	(SP, FP)	(FP, SP)	(FP, FP)
$(4, 0)$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$
$(3, 1)$	$\{0.04285, 0.03035\}$	$\{0.04285, 0.03035\}$	$\{0.04301, 0.03081\}$	$\{0.04063, 0.04063\}$
$(2, 2)$	$\{0.03951, 0.03951\}$	$\{0.02893, 0.04911\}$	$\{0.04911, 0.02893\}$	$\{0.04063, 0.04063\}$
$(1, 3)$	$\{0.03035, 0.04285\}$	$\{0.03081, 0.04301\}$	$\{0.03035, 0.04285\}$	$\{0.04063, 0.04063\}$
$(0, 4)$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$

Exchange 2

Exchange 1	SP	FP
SP	$f \cdot \{0.43, 0.16\}$	$f \cdot \{0.275, 0.32\}$
FP	$f \cdot \{0.48, 0.12\}$	$f \cdot \{0.459375, 0.153125\}$

- Segment 1 has three advertisers, Segment 2 has one advertiser, Segment 0 has no advertiser, and $\delta > 0.01408$.

Auction Formats

Number of Advertisers	(SP, SP)	(SP, FP)	(FP, SP)	(FP, FP)
$(4, 0)$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$	$\{0.04063, -\}$
$(3, 1)$	$\{0.04285, 0.03035\}$	$\{0.04285, 0.03035\}$	$\{0.04301, 0.03081\}$	$\{0.04063, 0.04063\}$
$(2, 2)$	$\{0.03951, 0.03951\}$	$\{0.02893, 0.04911\}$	$\{0.04911, 0.02893\}$	$\{0.04063, 0.04063\}$
$(1, 3)$	$\{0.03035, 0.04285\}$	$\{0.03081, 0.04301\}$	$\{0.03035, 0.04285\}$	$\{0.04063, 0.04063\}$
$(0, 4)$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$	$\{-, 0.04063\}$

Exchange 2

Exchange 1	SP	FP
SP	$f \cdot \{0.43, 0.16\}$	$f \cdot \{0.43, 0.16\}$
FP	$f \cdot \{0.48, 0.12\}$	$f \cdot \{0.459375, 0.153125\}$

As we can see, in all cases there is a unique equilibrium where both exchanges have a positive number of advertisers and are using first-price auctions. □